A robust risk-based tactical asset allocation

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Abstract. In this paper we define and compare different versions of robust, in the sense of Robust Optimization, and non robust portfolio selection models alternatively based on the use of different risk measures. This with the aim to take account of investors' asymmetric preferences in profits and losses together with the goal of having solutions less dependent on the parameter uncertainty. The empirical implementation considers the time series of the monthly prices of some representatives benchmarks in a time period characterized by a very particular set of financial events and therefore an ideal time to test the different portfolios strategies related to the alternative models. We show that the robust CVaR approach is preferable compared with the others and with the risk-free portfolio. The results can have very interesting applications in the field of the asset management industry.
Keywords. Coherent risk measures, Conditional Value at Risk, robust optimization, benchmarks.

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1 Introduction

The leading example of the classical portfolio selection models is the famous and Nobel-awarded Markowitz's model [27], [28], [29] based on a bi–criteria optimization scheme whose goal is to form a portfolio in which expected return is maximized while the portfolio return variance or standard deviation (volatility) is minimized.
Actually, models that maintain the same bi–criteria scheme and that differ each other for the chosen risk measure are largely implemented for their conceptual simplicity and useful applications. Recently, however, three relevant problems have been observed [30], [31], [33]:

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1. a large literature [18] evidenced that the return probability distribution is not necessarily symmetric: it can present asymmetries on the right or on the left in relation to specific asset classes (i.e. derivatives, hedge funds, small cap etc.);
2. empirical evidence [37] generally shows that unexpected losses are perceive more negatively than the benefits derived from unforeseen positive results; then, positive and negative deviations of portfolio returns from their means play an asymmetric role in investors utility that must be adequately considered [25], [26];
3. optimal solutions heavily depend on the levels and perturbations of the model parameters (moments) that are not known with certainty [20].

As to the first and second point, financial practice and related theory has shown increasing interest towards downside risk and quantile based measures, such as the Value at Risk (VaR). The success of VaR as a synthetic risk measure is due to its wide applicability (although under strong validity conditions within the trading and investment operations) as well as to its use in relation to the provisions known as Basel II. In any case, VaR has become a common way to evaluate the aggregate risk but, if it is studied in the framework of coherent risk measures [2], [3], it lacks reasonable properties, like sub–additivity and convexity, in the case of general loss distributions. Actually, this drawback entails inconsistency with the well known accepted principle of diversification as well as difficulties from the point of view of numerical tractability.
To overcome these problems, recent literature on portfolio selection focused on coherent risk measures and in particular on Conditional Value at Risk (CVaR or Tail–VaR); some authors [32], [34], [35] consider that this particular risk measure is the right objective to be minimized.
Although in literature it is possible to find different definition for this risk measure, in this paper we will refer to that given by Rockafellar and Uryasev in 2002 according to which the CVaR is the mean of the $\alpha-$tail distribution of the portfolio loss function and it quantifies the (extremes) tail losses of the return distribution. In this sense it is also called expected shortfall.
Rockafellar and Uryasev [34], [35] and also Pflug [32] proved that the CVaR is a coherent risk measure in general (referring to any returns probability distribution), taking into account the tail risk; furthermore this formulation makes it possible to minimize CVaR using the classical methods of Linear Programming (LP).
Another interesting aspect related to the previous CVaR definition is that by solving a simple convex optimization problem in one dimension it is possible to obtain simultaneously the $CVaR_\alpha$ and the VaR_α of a portfolio. This result is particularly important because it allows us to calculate the CVaR of a financial position without knowing in advance the value of the VaR.
The third weakness point of classical portfolio selection models is the unpleasant circumstance that optimal solutions heavily depend on financial parameters perturbations, particularly on the expected returns vector of the considered assets [31], [15], [24]. When the financial parameters values are affected by uncertainty,

they can be known only more or less approximately according to the particular market context and according to the used estimation procedure; clearly, this heavy dependence of the optimal solutions on parameters perturbations generates results for practical purposes that are not realistic.

In recent years, this feature has been dealt with through a new methodology introduced in the optimization literature under the name of Robust Optimization [4], [6], [7], [8], [9], [10], [5], [22], [23].

In the following, we will adopt this approach applying Robust Optimization to portfolio selection models that differ each other for the symmetric or asymmetric measures of risk used. In particular, i) we will minimize the risk for a given (minimum) portfolio return (we will indicate this problem as a direct model) and ii) we will maximize the portfolio return for a given (maximum) risk level (we will indicate this problem as an inverse model) both in a standard (non robust, in the sense of Robust Optimization) way, using punctual fixed values for the (uncertain) model parameters, and then following the Robust Optimization approach.

We will call the models obtained by the Robust Optimization reformulation as robust counterpart of the original (non robust) models and, respectively, as robust direct models and robust inverse models. In all cases uncertainty will be considered in the expected return vector of each financial asset and, referring to such uncertainty, we will obtain the robust solution for any of the portfolio selection model.

Note that, in general, the reformulation of the problems obtained following the Robust Optimization principles will be not linear even in the situation in which the original problem is linear [8], [23], [20] then requiring more sophisticated and time demanding solution techniques. An important feature of our analysis is that we are able to avoid this problem through the use of the Soyster's approach within the Robust Optimization [38] obtaining a robust reformulation of the portfolio selection problems for which standard minimization procedures are available.

The paper is structured as follows: in Section 2 we will introduce Robust Optimization, analyzing how risk and uncertainty emerging from parameters variability can be handled when the unknown parameters belong, with some confidence level, to certain intervals or variation ranges. In Section 3 we will discuss the possible geometries of the uncertainty set, closely related to the possibility to computationally solve the robust reformulation of the models.

In Section 4 we will illustrate the original (non robust) and the robust portfolio selection models to which we will refer in the implementation. In particular, in Section 5 we will implement, following the Robust Optimization schemes, six portfolio selection models obtained alternatively considering three different (symmetric and asymmetric) risk measures, such as Volatility (Vol), VaR and CVaR, as the objective function to be minimized (robust direct models) or as a constraint in the maximization problem of the portfolio's return (robust inverse models). With the aim to best evaluate and compare the advantages of using the Robust Optimization approach in portfolio selection, in the same section we

will also implement the non robust counterparts of the previous models, such as the previous quoted original models.
The comparison will be done through an out of the sample analysis of the results obtained by the ex-ante implementation of each model in selecting portfolios from a set of ten benchmarks from January 2007 to September 2009; note that this period of time has been characterized by a number of very particular financial events and therefore it is unquestionably an ideal time period to test the different portfolio strategies obtained by the different models.
As we shall see, the strategies obtained by means of the Robust Optimization approach have a definitely better performance and, among the robust models, the CVaR model dominates the other competitors because of its coherent nature.

2 Robust optimization models

In the real world, it is very difficult to find examples of systems that do not include some level of uncertainty about the values to assign to some or all of their parameters or about the actual design of some of their components. On the contrary, the optimization models are almost all formulated in a deterministic way assuming that the values to assign to the parameters are exactly known [42], [5].

It is true that in a number of cases not much is lost by assuming that these uncertain quantities are actually known, either because the level of uncertainty is low or because they play a less significant role in the process that must be analyzed or controlled.

But most frequently the uncertain parameters values play a central role in the analysis of the decision making process: an example can be the value to assign to the expected return of a financial asset at a future time T; in such a context the peculiarity of these parameters cannot be ignored without the risk of invalidating the possible implications of the analysis [39], [40], [30].

Some risk management models that consider uncertain parameters values based on the portfolio VaR or CVaR minimization belong to the Stochastic Programming (PS) problems [41], [34], [35], [36], [43][1]. However it was noted [6], [8], [14] that, generally, using a Stochastic Programming approach, constrains can be violated with a given probability; as a consequence, in a model of Stochastic Programming which faces data uncertainty, the variables do not necessarily satisfy the original constraints, but only a relaxed version of them. So it is generally accepted that a Stochastic Programming approach can mainly treat only the so called soft constraints.

Robust Optimization can overcome this problems assuming that the uncertain parameters are known to belong to particular value intervals or it is possible to suppose that, with a certain confidence level, the values can span particular variation ranges. So, the goal becomes to find a solution (called robust solution)

[1] As it is well known, every Stochastic Programming model can be viewed as an extension of a deterministic (linear or nonlinear) model where the uncertain parameters are given a probabilistic representation.

which is feasible for all possible data realizations (therefore it can satisfy hard constrains) and optimal in some respect.

In this section, we will take exclusive interest in the identification of robust solutions, in the sense of Robust Optimization, for Convex Programming models with uncertainty in the parameters; this choice is strictly functional to the applications that will be illustrated in Section 5.

Referring to the papers of Ben Tal and Nemirovski [6], [8], consider the following Convex Programming problem

$$\min_{x}\{f(\mathbf{x},\mathbf{c}), A\mathbf{x} \geq \mathbf{b}\}, \tag{1}$$

in which $\mathbf{x}$ and $\mathbf{c} \in R^N$, $A \in R^{n\times N}$, and $\mathbf{b} \in R^n$. The parameter $\mathbf{c}$ of the objective function $f(\mathbf{x},\mathbf{c})$ and the data $A, \mathbf{b}$ are not known exactly; what is known is a domain I in the space of data, an uncertainty set, which for sure contains the actual data and, in spite of this uncertainty, the decision vector $\mathbf{x}$ must satisfy for sure (whether it is possible to know them or not) the actual constraints.

We will call non robust (original) model the problem that consider as certain fixed values the uncertain parameters. On the contrary, we will call robust counterpart of the non robust (original) model or, more easily, robust model the model that try to consider such an uncertainty.

The only way to meet the requirements is to restrict ourselves to robust feasible candidate solutions, i.e. those which satisfy all possible realizations of the uncertain constraints. The admissible region is given therefore by

$$\{\mathbf{x}|A\mathbf{x} \geq b \;\; \forall [A,\mathbf{b}] : \exists \mathbf{c} : (\mathbf{c}, A, \mathbf{b}) \in I\}. \tag{2}$$

With the aim to choose the best among the feasible solutions, it is necessary to decide how to “aggregate” the different realizations of the objective into a single quality characteristic; in order to be methodologically consistent, a robust rule of min–max type is adopted in virtue of which it is minimized the maximum (worst case) on all the possible realizations of the objective function $f(\mathbf{x},\mathbf{c})$[2]; as a consequence, we have taken the following objective function

$$t = sup\{f(\mathbf{x},\mathbf{c})|\exists [A,\mathbf{b}] : (\mathbf{c}, A, \mathbf{b}) \in I\}. \tag{3}$$

In this way it is possible to associate to the original convex uncertain problem (1), or more precisely to the family of all certain convex models whose data belong to the uncertainty set I, its robust counterpart, in which the search for the smallest possible value of t takes the following form:

$$\min_{t,\mathbf{x}}\{t : t \geq f(\mathbf{x},\mathbf{c}), A\mathbf{x} \geq \mathbf{b} \; \forall (\mathbf{c}, A, \mathbf{b}) \in I\}, \tag{4}$$

where the component $\mathbf{x}^*$ of the solution vector $(\mathbf{x}^*, t^*)$ is called robust solution of the optimization model.

[2] There exist different methods to approach this problem that are widely implemented and compared in many engineering applications [1].

3 The geometry of the uncertainty set

The robust counterpart of the original model has actually the structure of an usual deterministic optimization problem but, having a continuum of constraints deriving from assessing the parameters values within their chosen variation range, it appears as a semi–infinite problem and, as such, it could seem computationally intractable.

The possibility to computationally solve it, as we shall illustrate, is closely and directly related to the geometry of the uncertainty set. In other words, if the goal is to find the best among the different feasible points of the robust counterpart, it is fundamental to specify the geometry of the uncertainty set I, because the structure of the robust counterpart of the original convex problem and the related possibility to computationally solve it will depend on this geometry.

Ben Tal and Nemirovski showed [6], [10] that this issue exclusively depends on the mathematical description of the uncertainty set I.

Theorem 1 (Nature of the robust counterpart of a LP problem). *Consider the following LP problem in which $\boldsymbol{x} \in R^N$ and the data $\boldsymbol{c}, A, \boldsymbol{b}$ are not known exactly*

$$\min_{x}\{f(\boldsymbol{x},\boldsymbol{c}) : A\boldsymbol{x} \geq \boldsymbol{b},\ (\boldsymbol{c}, A, \boldsymbol{b}) \in \mathrm{I} \subset R^N \times R^{n\times N} \times R^n\} \tag{5}$$

and its robust counterpart

$$\min_{t,\boldsymbol{x}}\{t : t \geq f(\boldsymbol{x},\boldsymbol{c}), A\boldsymbol{x} \geq \boldsymbol{b}\ \ \forall(\boldsymbol{c}, A, \boldsymbol{b}) \in \mathrm{I}\}. \tag{6}$$

Assume that the uncertainty set I *is given as the affine image of a bounded set* $Z = \{\zeta\} \subset R^M$ *and that* Z *is given either by*

1. *a system of linear inequality constraints* $P\zeta \leq p$, *or*
2. *a system of Conic Quadratic inequalities* $\|P_j\zeta - p_j\|_2 \leq e_j^T\zeta - d_j$ *in which* $j = 1, \ldots, L$ *and* $e_j \in R^L$, *or*
3. *a system of Linear Matrix Inequalities* $P_0 + \sum_{j=1}^{dim\zeta} P_j\zeta_j \geq 0$.

In the cases 1. *and* 2. *assume that the system of constraints that defines the uncertainty set* I *is strictly feasible.*
Then the robust counterpart of the original Linear Programming problem is respectively equivalent to

- *a Linear Programming problem,*
- *a Conic Quadratic Programming Problem (SOCP)*
- *a Semi–Definite Programming problem*[3].

The question is really very delicate. As noted by Ben Tal and Nemirovski [9], [10], [11] the computational tractability of the uncertainty set I is a theoretical property which is not exactly equivalent to the efficient solvability in practice,

[3] For the Proof of the quoted Theorem, please see [6], [10].

mainly for the large size of some real world convex problem as those used for portfolio selection.

In order to come up with a solution, it is highly desirable to ensure a simple analytical structure of the robust counterpart of the original problem, and this in turn requires the uncertainty set to be relatively simple. But when restricting ourselves to too simple geometries of the uncertainty set we certainly loose in flexibility of the approach.

So it is very important to find a valuable trade–off between a more realistic description of the problem, involving a complex structure of the uncertainty set I, and an its simple geometry, which entails less calculations and faster solutions particularly important in financial applications.

The Soyster's approach [38] can give a solution to the semi–infinite nature of the optimization problem providing a computationally fast solution.

The author, that was one of the first to discuss about hard uncertain constraints in Linear Programming models, considered a problem in which the N columns $(\mathbf{a}_i)$ of the matrix $A_{n\times N}$ are known to belong to given convex sets C_i.

In particular, the Linear Programming model considered by Soyster was the following

$$\max_x \{f(\mathbf{x}, \mathbf{c}), A\mathbf{x} \geq \mathbf{b}, \mathbf{x} \geq 0\}, \tag{7}$$

where $\mathbf{c}$ and $\mathbf{x} \in R^N$, $\mathbf{b} \in R^n$ and $A = (\mathbf{a}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{a}_N)$, with $\mathbf{a}_i \in R^n$.

If the vectors $\mathbf{a}_i$ are the estimates of the true parameters related to the variables, it is supposed that the true vector related to the $i-th$ parameter lies in the following hypersphere with center $\mathbf{a}'_i$ and radius $\mathbf{s}_i$, so

$$C_i = \{\mathbf{a} \in R^n : |\mathbf{a} - \mathbf{a}'_i| \leq \mathbf{s}_i\}, \tag{8}$$

where $\mathbf{s}_i$ measures the amount of uncertainty associated with $\mathbf{a}'_i$ [4]. Therefore, the original deterministic problem can be replaced by the following

$$\sup_{\mathbf{X}} \{f(\mathbf{x}, \mathbf{c}), x_1 C_1 + \ldots + x_N C_N \subseteq C(b), \mathbf{x} \geq 0\}, \tag{9}$$

in which there are N convex and non–empty sets C_i and a convex and non–empty set $C(b) = \{\mathbf{y} \in R^n : \mathbf{y} \leq \mathbf{b} \in R^n\}$.

In his paper Soyster proved that the previous problem can be solved via the following associated problem

$$\max_{\mathbf{X}} \{f(\mathbf{x}, \mathbf{c}), A^*\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq 0\}, \tag{10}$$

where a^*_{ji} (general element of the matrix A^*) is a number given by $\sup_{\mathbf{a}_i \in C_i} \mathbf{a}_{ji}$; the optimal solutions of the two problems are in fact the same.

But when the sets C_i are hyperspheres, using the canonical vectors $\mathbf{e}_j$, we can write that

[4] Obviously, $\mathbf{s}_i = 0$ if the $i-th$ vector is exactly known.

$$\sup_{\mathbf{a}\in C_i} \mathbf{e}_j \cdot \mathbf{a} = \mathbf{a}'_{ji} + \mathbf{s}_i, \quad \forall j, \tag{11}$$

so the vectors of the matrix A^* with general elements $\mathbf{a}^*_{ji}$ will be $\mathbf{a}^*_i = \mathbf{a}'_i + \mathbf{s}_i \cdot \mathbf{1}$, where $\mathbf{1} = (1, ..., 1)^T$.

Hence, the optimal solution of the problem (9) can be determined by solving the easier one

$$\max_{\mathbf{x}} \{f(\mathbf{x}, \mathbf{c}), x_1(\mathbf{a}'_1 + \mathbf{s}_1 \cdot \mathbf{1}) + \ldots + x_N(\mathbf{a}'_N + \mathbf{s}_N \cdot \mathbf{1}) \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}. \tag{12}$$

According to this point of view, the constraints of problem (4) are completely equivalent to a system of linear constraints. So, the semi–infinite robust counterpart (4) of the original optimization problem (1), via the Soyster's approach coherently with Theorem 1 becomes a convex problem and, as a consequence, it can be easily and quickly solved.

As it was noted [4], [8], [12], [13], if the Soyster's approach is considered from a financial point of view, it appears very conservative assuming that each element of the constraint matrix of the problem is the worst it could be while, in general, and particularly if we consider the assets returns, the coefficients are not on their worst values at the same time.

Ben Tal and Nemirovsky [10] proposed a less conservative definition of the uncertainty set (see also [6], [7], [20]) showing that for their ellipsoidal uncertainty set the robust counterpart of a linear, quadratic or, in general, convex problem is an optimization model solvable via the interior points method.

But under the hypothesis that the uncertainty sets have such a geometry, the robust counterpart of a problem of any original nature, even if it is very simple as a linear programming problem, becames a non linear model with huge computational difficulty. Moreover, excluding the case of a linear programming problem, when the uncertain coefficients follow special probability distribution, no certainty exists that the obtained robust solution is feasible [14], [12], [13].

So, Soyster's approach, although introduced almost forty years ago, seems still to be the best choice to obtain a realistic (even if rather prudential) and easily solvable description of the problem linked with a sufficiently simple geometry of the uncertainty set.

4 Robust CVaR, VaR and Vol optimization models

In this paper we want to implement, in the sense of Robust Optimization, the CVaR, the VaR and the Volatility (Vol) minimization model (direct models) and the portfolio return maximization models that used the previous measures as a risk constraint (inverse models).

We start from the following general form of the (original) non robust direct model

$$\min_{\mathbf{x}}\{f(\mathbf{x}), \mathbf{r}^T\mathbf{x} \geq g, \mathbf{1}^T\mathbf{x} = 1\}, \tag{13}$$

and the non robust inverse model

$$\max_{\mathbf{x}}\{\mathbf{r}^T\mathbf{x}, f(\mathbf{x}) \leq R, \mathbf{1}^T\mathbf{x} = 1\}, \tag{14}$$

where $\mathbf{x} \in R^N$ is the vector of the portfolio weights, $f(\mathbf{x})$ represents the risk measure considered, g is a scalar that quantifies the minimum return required to the optimum portfolio, R is a suitable risk threshold and $\mathbf{r} \in R^N$ is the assets expected return vector. Note that $\mathbf{r}$ is assumed known in these models.

Instead, assuming uncertainty in vector $\mathbf{r}$[5], we obtain, as it was discussed in Section 2, the following general form of the robust direct model

$$\min_{\mathbf{x}}\{f(\mathbf{x}), \min_{\mathbf{r}\in I}\{\mathbf{r}^T\mathbf{x} \geq g\}, \mathbf{1}^T\mathbf{x} = 1\}, \tag{15}$$

and the robust inverse model

$$\max_{\mathbf{x}}\{\min_{\mathbf{r}\in I}\{\mathbf{r}^T\mathbf{x}\}, f(\mathbf{x}) \leq R, \mathbf{1}^T\mathbf{x} = 1\}, \tag{16}$$

where the constraint $\mathbf{r} \in I$ indicates that the expected assets returns are not fixed values but levels that can vary within particular variation intervals described by the uncertainty set I.

Then, if we assume that the parameters in the vector $\mathbf{r}$ are known to belong within particular ranges described by the confidence intervals of the parameters estimates, in accordance to the Soyster's approach it is possible to define the uncertainty set as $I = \{\mathbf{r}|\mathbf{r}' - \mathbf{s} \leq \mathbf{r} \leq \mathbf{r}' + \mathbf{s}\}$, where $\mathbf{r}'$ is the vector containing the estimates of the assets expected return while $\mathbf{s}$ is the vector collecting the standard deviation of such estimates. As a consequence, the robust semi–infinite counterparts of the considered models became computationally tractable because it is also possible to show [22], [23], [33] that we have

$$\min_{\mathbf{r}\in I} \mathbf{r}^T\mathbf{x} = \mathbf{r}'^T\mathbf{x} - \mathbf{s}^T|\mathbf{x}|. \tag{17}$$

Introducing a new variable $\mathbf{m}$ to replace $|\mathbf{x}|$ and adding the constraints $\mathbf{m} \geq \mathbf{x}$ and $\mathbf{m} \geq -\mathbf{x}$, it is possible to obtain the following general form of the robust direct model

$$\min_{\mathbf{x}}\{f(\mathbf{x}), \mathbf{r}'^T\mathbf{x} - \mathbf{s}^T\mathbf{m} \geq g, \mathbf{m} \geq \mathbf{x}, \mathbf{m} \geq -\mathbf{x}, \mathbf{1}^T\mathbf{x} = 1\} \tag{18}$$

and the robust inverse model

[5] Note that the expected asset return is a very difficult parameter to evaluate; moreover, its impact on the portfolio optimal allocation $\mathbf{x}^*$ is very high. A number of models in which uncertainty is referred both to this parameter and to other components of the portfolio selection model can be found in [23].

$$\max_{\mathbf{x}}\{\mathbf{r}^{'T}\mathbf{x}-\mathbf{s}^T\mathbf{m}, f(\mathbf{x})\leq R, \mathbf{m}\geq \mathbf{x}, \mathbf{m}\geq -\mathbf{x}, \mathbf{1}^T\mathbf{x}=1\}^6. \quad (19)$$

Now, using different risk measures, we obtain the following three robust direct models:

$$\begin{gathered}\min_{a,\mathbf{x}}\{CVaR_\alpha(-\mathbf{r}^T\mathbf{x})=a+\tfrac{1}{1-\alpha}\mathbf{E}(-\mathbf{r}^T\mathbf{x}-a)^+, \mathbf{r}^{'T}\mathbf{x}-\mathbf{s}^T\mathbf{m}\geq g,\\ \mathbf{m}\geq \mathbf{x}, \mathbf{m}\geq -\mathbf{x}, \mathbf{1}^T\mathbf{x}=1\}\end{gathered} \quad (20)$$

$$\begin{gathered}\min_{\mathbf{x}}\{VaR_\alpha(-\mathbf{r}^T\mathbf{x})=\mathbf{Pr}(-\mathbf{r}^T\mathbf{x}\leq\beta)\leq\alpha, \mathbf{r}^{'T}\mathbf{x}-\mathbf{s}^T\mathbf{m}\geq g,\\ \mathbf{m}\geq \mathbf{x}, \mathbf{m}\geq -\mathbf{x}, \mathbf{1}^T\mathbf{x}=1\}\end{gathered} \quad (21)$$

$$\min_{\mathbf{x}}\{Vol=\sqrt{(\mathbf{x}^T\Sigma\mathbf{x})}, \mathbf{r}^{'T}\mathbf{x}-\mathbf{s}^T\mathbf{m}\geq g, \mathbf{m}\geq \mathbf{x}, \mathbf{m}\geq -\mathbf{x}, \mathbf{1}^T\mathbf{x}=1\}, \quad (22)$$

where α is the threshold chosen to individuate the VaR, $-\mathbf{r}^T\mathbf{x}$ is the portfolio loss function obtained via historical simulation and typically defined as the negative of the portfolio return function, a is the value of the VaR in the optimal solution, $a^+=\max\{a,0\}$ and $\Sigma=[\sigma_{ij}]$, with $i,j=1,...,N$, (symmetric) positive definite matrix of $N\times N$ dimension is the variance-covariance matrix of the (historical) returns of the assets[7]. We have also the following three robust inverse models:

$$\begin{gathered}\max_{\mathbf{x}}\{\mathbf{r}^{'T}\mathbf{x}-\mathbf{s}^T\mathbf{m}, CVaR_\alpha(-\mathbf{r}^T\mathbf{x})=a+\tfrac{1}{1-\alpha}\mathbf{E}((-\mathbf{r}^T\mathbf{x}-a)^+)\leq R,\\ \mathbf{m}\geq \mathbf{x}, \mathbf{m}\geq -\mathbf{x}, \mathbf{1}^T\mathbf{x}=1\}\end{gathered} \quad (23)$$

$$\begin{gathered}\max_{\mathbf{x}}\{\mathbf{r}^{'T}\mathbf{x}-\mathbf{s}^T\mathbf{m}, VaR_\alpha(-\mathbf{r}^T\mathbf{x})=[(\mathbf{Pr}(-\mathbf{r}^T\mathbf{x}\leq\beta))\leq\alpha]\leq R,\\ \mathbf{m}\geq \mathbf{x}, \mathbf{m}\geq -\mathbf{x}, \mathbf{1}^T\mathbf{x}=1\}\end{gathered} \quad (24)$$

$$\max_{\mathbf{x}}\{\mathbf{r}^{'T}\mathbf{x}-\mathbf{s}^T\mathbf{m}, Vol=\sqrt{(\mathbf{x}^T\Sigma\mathbf{x})}\leq R, \mathbf{m}\geq \mathbf{x}, \mathbf{m}\geq -\mathbf{x}, \mathbf{1}^T\mathbf{x}=1\}. \quad (25)$$

It is possible to quickly obtain the solutions of all these models with very low computational costs.

In Section 5 we will implement these six robust models and their non robust counterpart obtained from (13) and (14) and using for $f(\mathbf{x})$ the same expressions of the CVaR, the VaR and the Vol from (20) to (25).

[6] Note that we not exclude here the possibility of short selling because we prefer to illustrate the more general structure of the robust counterparts. On the contrary, in the implementation of the models given in Section 5 we will include the constraint $\mathbf{x}\geq 0$.

[7] As it is well known, in such a matrix, the N variances of the (historical) returns referred to the N considered assets are in the main diagonal while the $\frac{N(N-1)}{2}$ covariances are the elements of the matrix (triangular) upper part and, for symmetry, of its lower part.

5 Computational results

In this section we illustrate the results obtained implementing Non Robust and Robust Optimization in order

1. to minimize the CVaR, the VaR and the Vol of a portfolio composed by ten representative benchmarks in relation to an established minimum portfolio return threshold (direct models) and
2. to maximize the expected return of the portfolios composed by the same ten assets satisfying a risk constraint that considers the previous risk measure having values lower than those of a fixed risk threshold (inverse model).

The aim is to individuate an optimal selection strategy for dynamic portfolio rebalancing. In particular, we implement the robust models from (20) to (25) and their non robust counterpart making some comparisons that can help to evaluate both the Robust Optimization advantages in the asset management field and the benefits deriving from the alternative different risk measures.

Note that uncertainty is referred to the vector $\mathbf{r}$ containing the considered assets expected return; it is surely the more difficult parameter to evaluate and it has a very high impact on the optimal solutions.

In the optimization problems referred to the CVaR we follow Rockafellar and Uryasev [34]; in the VaR optimization problems the risk measure is modelled as suggested by Gaivoronski and Pflug [21] while the variance, and then the volatility, is obtained by the usual historical estimation procedure.

In the empirical implementation we considered the time series from January 2007 to September 2009 of the monthly prices of some representatives benchmarks. In particular we used the Dow Jones Euro Stoxx 50, SP 500, Topix, MSCI Pacific ex Japan, MSCI Emerging Markets, FTSE Epra–Nairet Dev. Europe, JP Morgan EGBI, Barclays G7 Global Bonds, JP Morgan Emerging Bonds and JP Morgan Euro Cash 3m.
Note that the considered period of time has been characterized by a sequence of very relevant financial events; we want to remember the different market regimes, with a significative stock markets increase in the first and in the last six month period and, on the contrary, the huge stock exchange collapses (Real Estate and Credit crisis respectively in 2007 and 2008) and volatilities that reached record levels at the end of 2008.

It is an unquestionably ideal time period to conduct a back-test of the portfolio strategies obtained by the different models and to compare the predictive efficiency of the considered risk measures.

The market costs related to each transaction required by portfolio rebalancing have been considered as a fixed fee of 4/1000.

Using GAMS[8] we wrote (i) a code for the minimization of the risk measures (direct models) using as minimum threshold (g) of the monthly portfolio return the value at the beginning of every month of the $1-$month Libor rate as the closest approximation of a riskless asset in the market; (ii) a code for

[8] General Algebraic Modelling System, Gams.Ide Gams Rev. 135 Vis. 21.1 135 [16]

the maximization of the portfolio return (inverse models) using as maximum risk threshold (R) a value equal to 3% in term of volatility and its gaussian equivalent for VaR and CVaR.

The confidence level chosen for the quantification of the CVaR and VaR has been fixed at 99%.

Because of their random nature and their crucial role within the models, special attention has been paid to an efficient evaluation of the values $\mathbf{r}'$ estimated for the period from January 2007 to September 2009 using the monthly data of the considered benchmarks and a Multivariate Vector Autoregressive Model with lagged endogenous variables and, as exogenous, the exchange rates dollar/euro and yen/euro[9]. In particular, a fixed historical window of the last four years of monthly data was used each month obtaining, at every recursion, the 1 month ahead prediction of the monthly return of each benchmark.

The lower and upper limits $\mathbf{r}' \pm \mathbf{s}$ of the uncertainty set were obtained as the extremes values of a 60% confidence intervals. It may be interesting to note that the actual (true) returns of each considered benchmarks, calculated at time $(t+1)$ always lie within the confidence intervals estimated at time t [10]; this circumstance is particularly important for the construction and implementation of the robust counterparts of the portfolio selection models. At the end of month t, considering the predictions and the confidence intervals for $(t+1)$, the portfolio is optimized (out-of-sample results) according to the various models. The new portfolios $\mathbf{x}_{t+1|t}$, rebalanced and then taking into account transaction costs, will produce gains or losses at the end of month $(t+1)$; the process is rolled over for month $(t+2)$ and so on.

In Table 1 and in Fig. 1 to Fig. 5 we summarize the results obtained in this implementation. Given that direct and inverse models gave very similar results, in Table 1 we report only those achieved by the inverse models that are easier to translate in financial products with a given risk profile [18].
The results show that:

- in the three years from 2007 to 2009 the risk–free portfolio obtained by a roll-over on the Libor rate reached a return of 9.46%. Such a value has been overcome only by the net return of the robust CVaR model (15.44%). Actually, both the robust and the non robust CVaR models are those related to the higher turnover index and therefore are more penalized by the transaction costs incidence. In any case, they are able to reach the higher returns.
- as it can be noted if Fig. 1, the non robust portfolios registered negative performances for many months. Moreover the performance of the non robust CVaR model has been generally better than those of the other two models; the CVaR return was 0.32%. The robust CVaR model, net of transaction costs, obtains the highest total return (15.44%);

[9] To this purpose, we used SAS 8.2.
[10] The variance–covariance matrixes of the historical returns have been calculated each month and are available upon request

Table 1. Ex–post comparisons of portfolios selected by different inverse models (Long–only: $\mathbf{x} \geq \mathbf{0}$).

	Models	Non–Robust	Robust
Portfolio's Returns	VaR	-20.58	-1.85
	CVaR	0.32	15.44
	Vol	-4.07	7.51
	Risk–free	9.46	9.46
Volatility of the Portfolio's Returns	VaR	9.93	5.62
	CVaR	8.32	6.97
	Vol	5.64	3.55
	Risk–free	0.44	0.44
VaR of the Portfolio's Returns	VaR	-36.30	-12.97
	CVaR	-18.91	-9.54
	Vol	-17.96	-5.92
	Risk–free	0.13	0.13
CVaR of the Portfolio's Returns	VaR	-46.53	-13.96
	CVaR	-22.81	-9.91
	Vol	-22.81	-6.10
	Risk–free	0.13	0.13
Asymmetry of the Portfolio's Returns	VaR	-2.81	-0.17
	CVaR	-0.01	0.43
	Vol	-1.73	0.66
	Risk–free	-0.93	-0.93
Kurtosis of the Portfolio's Returns	VaR	12.06	0.47
	CVaR	1.36	-0.45
	Vol	7.36	1.54
	Risk–free	-0.84	-0.84

- the ex–post volatilities of the actual returns of the robust models are lower than those of the corresponding non robust models; the higher value is that of the CVaR model (6.97%) showing that the volatility is not always a good risk measure;
- the robust CVaR and Vol models have positive asymmetry's values[11] of the ex–post returns (respectively, 0.43 and 0.66). This means that the portfolio's management has been effective in positively deforming the probability distributions and then in moving the probability towards the right–hand side; on the contrary, all the other models have a negative asymmetry;
- the (excess) kurtosis [12] of the returns of the non robust models is always positive with upper value registered by the VaR model (12.06) and lower value by the CVaR model (1.36); in the robust approach, the kurtosis of the

[11] The asymmetry is calculated as third moment, centered with respect to the mean, divided by the third power of the standard deviation.

[12] The excess kurtosis is calculated as the distance from 3 (the normal kurtosis) of the forth moment, centered with respect to the mean, divided by the squared of the variance.

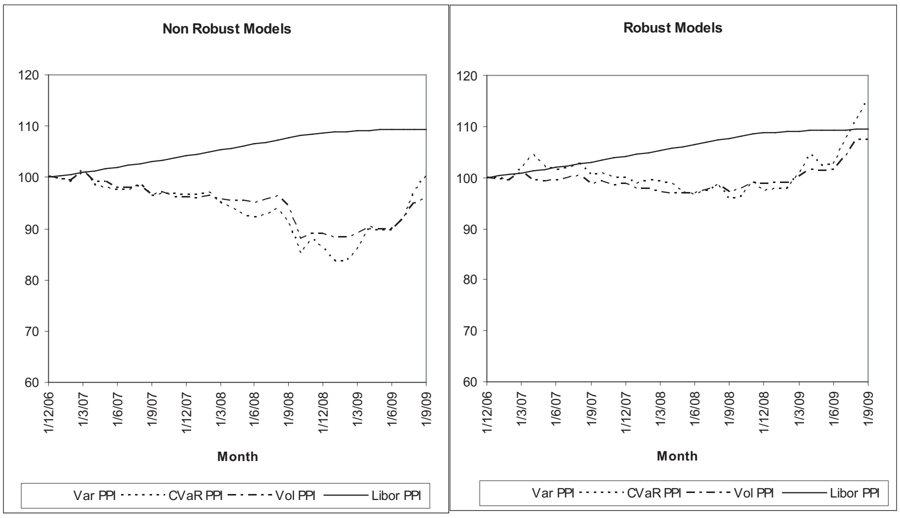

Fig. 1. Comparisons among portfolio performance indexes obtained by different Non Robust (on the left) and Robust (on the right) models.

CVaR model is negative (−0.45), indicating a value under the normal one. This means that for the CVaR the frequency of the extreme events (in the “wrong” tail) is, ex–post, lower with respect to that of normal type. In the robust approach, the higher value is that of the Vol model (1.54) that, in any case, is almost five time lower than that of the non robust case[13] suggesting non rejection;

– the robust and non robust CVaR approaches, always provided a number and an amount of losses lower than those obtained by the robust and non robust models that use the VaR or the Vol as risk measure (Fig. 2);

– the ex–post analysis also shows that the CVaR model, especially in its robust version, always obeys the risk constraint imposed in the portfolio optimization (Fig. 3). The same is not true for the VaR (Fig. 4) and for the Vol (Fig. 5) models in relation to which it is possible to note, especially in relation to their non robust version, a frequent ex–post violation of the maximum risk threshold ex–ante imposed.

[13] An asyntotic inferential test of normality [19] has been conducted. This test, with a confidence level of 95% allows to reject for the Vol model the null hypothesis of normality of the returns. For the VaR and CVaR models the marginal significativity was, respectively, of 56% and of 50.9%,

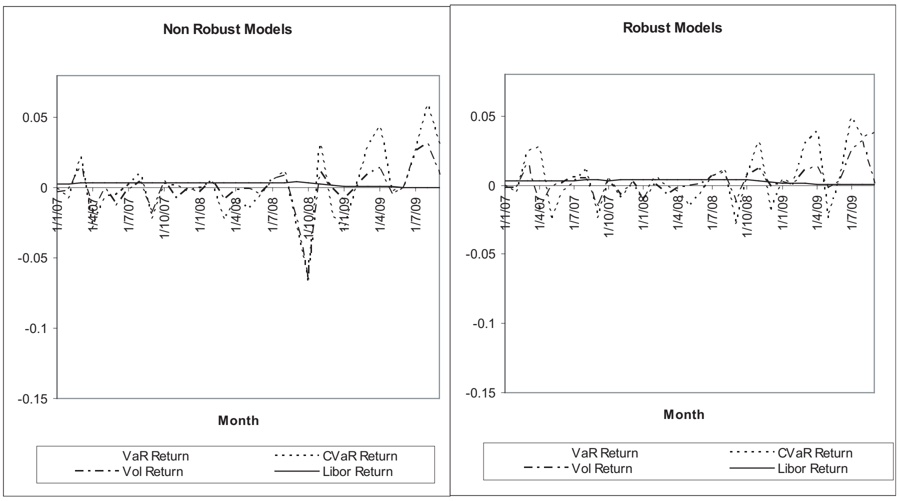

Fig. 2. Comparisons among the actual portfolio monthly returns obtained by different Non Robust (on the left) and Robust (on the right) models.

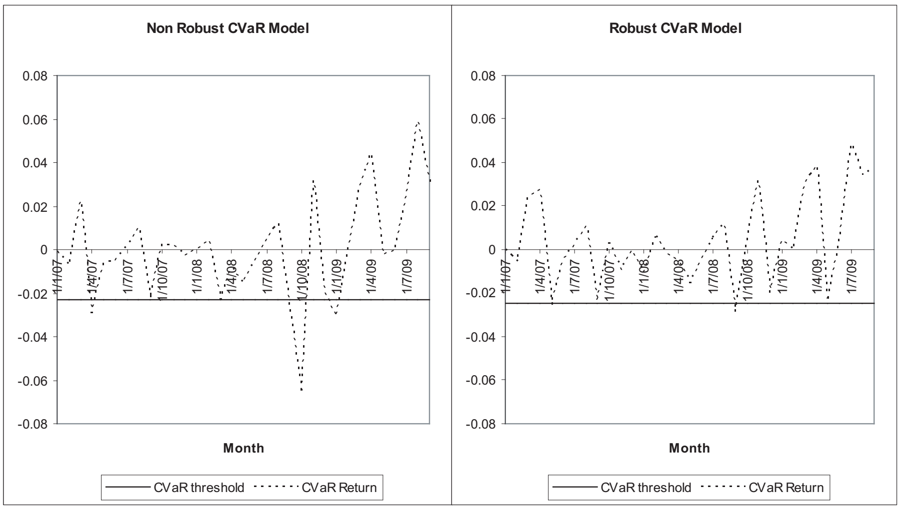

Fig. 3. Back–test: CVaR threshold and returns obtained by the Not Robust (on the left) and Robust (on the right) CVaR model.

6 Conclusions

In this paper we implemented in the sense of Robust Optimization a particular portfolio selection approach that is able to consider the uncertain values as-

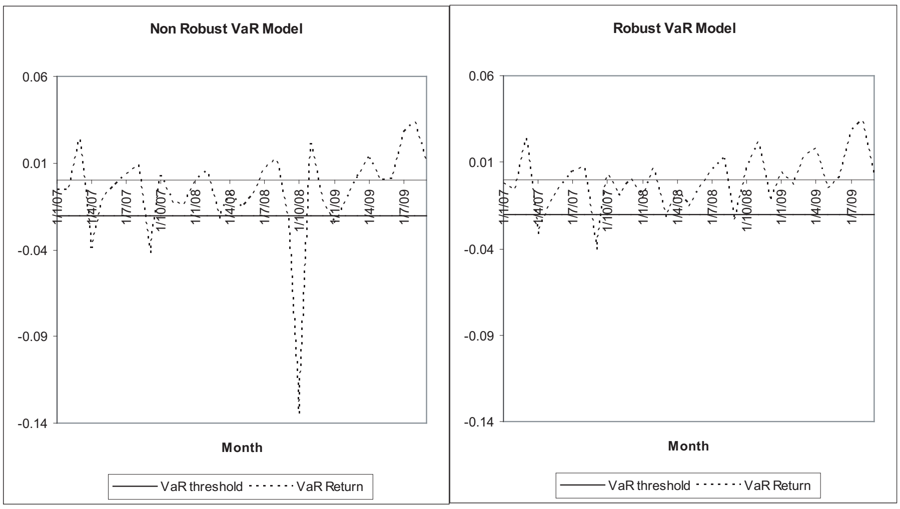

Fig. 4. Back–test: VaR threshold and returns obtained by the Not Robust (on the left) and Robust (on the right) VaR model.

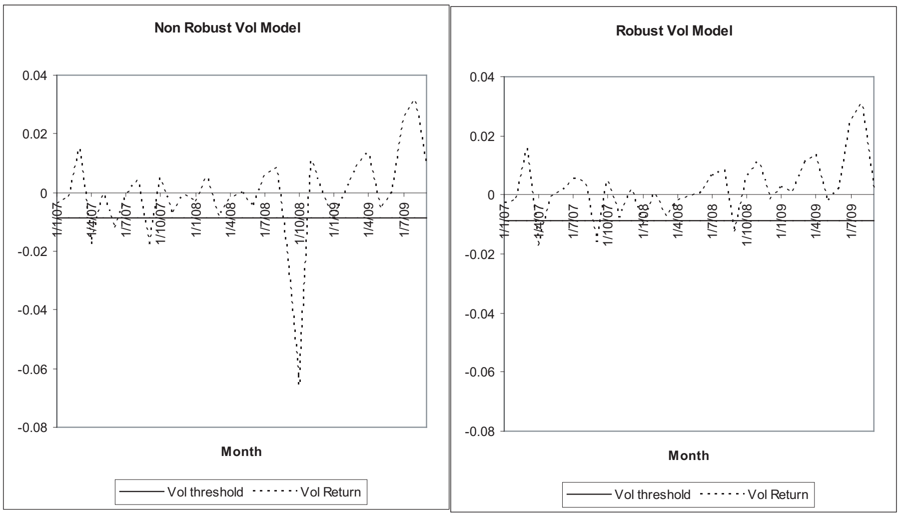

Fig. 5. Back–test: Vol threshold and returns obtained by the Not Robust (on the left) and Robust (on the right) Vol model.

sumed by the parameters within the optimization processes and, simultaneously, the asymmetry in the financial assets returns distributions and the asymmetric effects played on the investor utility function by profits and losses.

The second and the third feature have been considerated evaluating the characteristics and the performance of three different risk measures, such as the CVaR, the VaR and the Vol, while the uncertainty about the parameters value within the different models has been dealt with the consideration of a robust approach in the sense of the optimization of worst cases accordingly to the schemes of Robust Optimization together with the Soyster's approach about modelling the uncertainty set.

The implementation on some benchmark data in the period between January 2007 and September 2009 allows to verify how the defined robust models are to be preferred to their traditional non robust counterparts and, in particular, to what extent the robust CVaR model is able to give the best results in terms of return, asymmetry, kurtosis and risk level in the ex post comparison.

The robust CVaR approach to portfolio selection turned out to be highly better than all the competing non robust and robust models and also with respect to the risk–free portfolio obtained as a continuum investment in a current account; this result could surely have very interesting applications in financial engineering and asset management industry.

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