Convex comparison of minimal divergence martingale measures

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Abstract. We refine some criteria for the convex comparison of martingale densities suggested in Franke et al. [11] and Bellini and Sgarra [4]. We give sufficient conditions for comparison based on the classical notion of comparative convexity. We apply these conditions to the case of minimal f-divergence martingale measures, establishing an ordering result in the case of power divergences. We discuss the extension of the comparison to a multiperiod setting and provide several numerical examples.

Keywords. Convex comparison, relative convexity, Esscher martingale measure, minimal entropy martingale measure, minimal divergence martingale measure.

M.S.C. classification. 91G20, 60E15.
J.E.L. classification. C02, G10.

1 Introduction

The payoffs of the most common financial derivatives, such as European call or put options, are convex functions of the underlying asset. It is then very natural to introduce a convex order relationship between equivalent martingale measures (EMMs henceforth):

$$Q_1 \leq_{cx} Q_2 \text{ if } E_{Q_1}[h(S_T)] \leq E_{Q_2}[h(S_T)], \text{for each convex payoff } h(S_T);$$

that is, Q_2 assigns higher prices to each convex payoff. Since the expected value of the underlying is equal to the forward price under all EMMs, the necessary condition for convex ordering is always automatically satisfied.

Mathematical Methods in Economics and Finance – m^{2}ef
Vol. 7, No. 1, 2012

It is not easy to trace back the first reference in which the idea of convex ordering of EMMs appears explicitly in the financial literature; it is surely present in several papers either related to option pricing bounds or to the comparison of option prices in different models. For example, El Karoui et al. [10] compared option prices in a stochastic volatility model with a corresponding local volatility model; Møller [24] compared several martingale measures in the context of compound Poisson processes; Bergenthum and Rüschendorf [5] provided very general criteria for the convex comparison of semimartingales, although their results do not apply in the discrete time case; Bergenthum and Rüschendorf [6] treated in full details the case of Lévy processes, proving comparison results based on the Lévy characteristics. In the context of stochastic volatility models, Henderson [18] and Henderson et al. [19] provided an ordering result for p-optimal martingale measures.

In this paper we focus on minimal divergence martingale measures Q_f (MDMMs henceforth), that are defined as the minimizers of an f-divergence functional over the set $\mathcal{M}$ of EMMs:

$$Q_f = \underset{Q \in \mathcal{M}}{\arg\min} \, H_f(Q, P),$$

with

$$H_f(Q, P) = E_P \left[f \left(\frac{dQ}{dP} \right) \right],$$

where $f : (0, +\infty) \to (0, +\infty)$ is strictly convex and $f(1) = 0$.

From an intuitive point of view, $H_f(Q, P)$ represents a sort of pseudodistance between Q and P; indeed if $Q = P$, then $H_f(Q, P)$=0. The MDMMs are the martingale measures that are closest to P with respect to $H_f(Q, P)$.

The class of MDMMs has been thoroughly studied in the financial literature; it includes as special cases the minimal entropy martingale measure (MEMP), the minimal reverse entropy, the minimal variance, the p-optimal and the minimal Hellinger martingale measures. A general treatment of the necessary and sufficient conditions for the existence of MDMMs in semimartingale models can be found in [15]. As it is well known, the problem of divergence minimization arises naturally as the dual of a utility maximization problem (see [15], [3] or [9] and references therein).

We start in a one period setting by refining comparisons results based on the elasticity of the EMM density that have been suggested by Franke et al. [11] and Bellini and Sgarra [4]. In particular, we show how the classical notion of relative convexity (extensively applied in the financial economics literature by Arrow and by Pratt [28]) can be used to provide sufficient conditions for the convex comparison of EMMs. In [4] we applied similar results to show that the Esscher martingale measure introduced in [13] and the MEMP introduced in [12] are always comparable, and which one is dominating depends on the sign of the risk premium of the underlying asset. As a more general example, we consider here the so called *second-order* Esscher martingale measures introduced in [25] and show that they are also an ordered family.

We then show that in the special case of power divergences, defined by:

$$f_\alpha(t) = \begin{cases} t\ln t - t + 1 & \alpha = 1 \\ -\ln t + t - 1 & \alpha = 0 \\ \frac{t^\alpha - \alpha(t-1) - 1}{\alpha(a-1)} & \alpha \neq 0,\, \alpha \neq 1 \end{cases}$$

the corresponding MDMMs are always ordered and $\alpha_1 \leq \alpha_2 \Longrightarrow Q_{f_{\alpha_2}} \leq_{cx} Q_{f_{\alpha_1}}$. A similar result has been obtained by Henderson [18] and Henderson et al. [19] in the context of stochastic volatility models.

Finally we discuss the extension of this comparison to a multiperiod setting. The main problem is that in general the density of a MDMM cannot be written as the product of one period divergence-minimizing densities, so the extension of the comparison result from the one period to the multiperiod case is not trivial. However, following [17] or [14], it is quite natural to consider *local* MDMM, that *by construction* are defined as the product of one period divergence-minimizing changes of measure. By applying the same procedure of [4], we show that *local* power MDMMs are ordered as in the one period case.

The paper is structured as follows: in Section 2 we discuss criteria for convex comparison in the one period case, in Section 3 we apply them to the case of power MDMMs, in Section 4 we discuss multiperiod extensions and provide numerical examples.

2 Convex comparison of martingale measures: one period case

We consider a one period model with two assets: the risky asset has initial price $S_0 > 0$ and terminal random price $S_T > 0$, while the riskless asset has initial price B_0 and terminal price $B_0 e^{rT}$, where $r > 0$ is the riskfree rate. In this elementary model there are no arbitrage opportunities if and only if:

$$P(S_T > S_0 e^{rT}) > 0 \text{ and } P(S_T < S_0 e^{rT}) > 0,$$

and this is equivalent to the existence of a probability measure $Q \sim P$ such that:

$$E_Q[S_T] = S_0 e^{rT}. \tag{1}$$

Any probability measure $Q \sim P$ satisfying (1) will be called an equivalent martingale measure (EMM henceforth). Since the great majority of European payoffs are convex functions of the underlying price at maturity S_T, it is very natural to introduce the notion of convex ordering of martingale measures:

Definition 1. *We say that* $Q_1 \leq_{cx} Q_2$ *if*

$$E_{Q_1}[h(S_T)] \leq E_{Q_2}[h(S_T)]$$

for each convex function $h : [0, +\infty) \to \mathbb{R}$.

We have that $Q_1 \leq_{cx} Q_2$ if the price attributed by Q_2 to each convex payoff is greater or equal to the price under Q_1. In particular, under Q_2 the prices of all European call and put are higher or equal than the corresponding prices under Q_1. Assuming that the EMMs Q_1 and Q_2 are absolutely continuous with respect to P, we introduce the Radon-Nikodym densities:

$$\varphi_1(s) = \frac{dQ_1}{dP}, \qquad \varphi_2(s) = \frac{dQ_2}{dP}$$

that satisfy:

$$\begin{aligned} &\varphi_1, \varphi_2 \in L^1\left([0, +\infty), P\right), \quad \varphi_1, \varphi_2 \geq 0 \quad \text{P-a.s.,} \\ &\int_0^{+\infty} \varphi_1(s) dP(s) = \int_0^{+\infty} \varphi_2(s) dP(s) = 1, \\ &\int_0^{+\infty} s\varphi_1(s) dP(s) = \int_0^{+\infty} s\varphi_2(s) dP(s) = S_0 e^{rT}. \end{aligned} \tag{2}$$

All the standard results about convex order that can be found in [26] or [29] do apply; in particular, we have the well known characterization:

Proposition 1. *Let φ_1 and φ_2 be as in (2); then $Q_1 \leq_{cx} Q_2$ if and only if for each $K > 0$ we have $\int_0^K \int_0^t \varphi_1(s) dP(s) dt \leq \int_0^K \int_0^t \varphi_2(s) dP(s) dt$.*

An important sufficient condition for convex ordering is the *cut criterion*, that goes back at least to [22]. In order to state it properly, we recall the definition of the number of cuts between two functions (see for example [29]):

Definition 2. *Let φ_1 and φ_2 be as in (2); we say that φ_1 and φ_2 cut n times if there exist a partition $\mathcal{P} = \{I_1, I_2, ..., I_{n+1}\}$ of $[0, +\infty)$ into disjoint intervals I_k such that:*
i) $P(I_k) > 0$,
ii) $\varphi_2 - \varphi_1$ is not P-a.s. equal to 0 and has a constant sign on each I_k,
iii) $\varphi_2 - \varphi_1$ changes sign from each interval I_k to the next.
The alternating sequence of signs of $\varphi_2 - \varphi_1$ on the intervals I_k is called the sign sequence of $\varphi_2 - \varphi_1$.

The cut criterion is the following (see [11] and [4]):

Proposition 2. *Let φ_1 and φ_2 be as in (2); then φ_1 and φ_2 cut at least two times. If φ_1 and φ_2 cut two times with sign sequence +,−,+, then $Q_1 \leq_{cx} Q_2$.*

In many models the densities φ_i are smooth functions; in this case it is possible to define their elasticities as

$$\eta_i(s) = -s \frac{\varphi_i'(s)}{\varphi_i(s)} = -s \frac{d}{ds} \ln \varphi(s). \tag{3}$$

Franke et al. [11] proved a sufficient condition for convex ordering based on the elasticities η_i:

Proposition 3. *Let φ_1 and φ_2 be as in (2) and decreasing. Let η_1 and η_2 be as in (3). If η_1 is constant and η_2 is decreasing, then $Q_1 \leq_{cx} Q_2$.*

In the Black-Scholes model the pricing density is a power function of the price of the underlying at maturity and hence has constant elasticity, so Proposition 3 can be applied to assess underpricing or overpricing with respect to the Black-Scholes model. A similar comparison result, based on the semi-elasticities instead of the elasticities, can be found in the actuarial literature in [20].

Bellini and Sgarra [4] proposed a generalization of the criterion of Franke et al. [11]; I'm presenting here a slightly more general formulation, based on essentially the same ideas:

Proposition 4. *Let $\varphi_1, \varphi_2 \in C^1(0,+\infty)$ be as in (2), with $\varphi_1 \neq \varphi_2$. If any of the following conditions hold, then φ_1 and φ_2 cut exactly two times, and hence Q_1 and Q_2 are comparable in the convex order:*
i) $\varphi_2 - \varphi_1$ is convex or concave,
ii) $\varphi_2' - \varphi_1'$ is monotone,
iii) the ratio $\frac{\varphi_2'}{\varphi_1'}$ is monotone,
iv) the ratio of elasticities $\frac{\eta_2}{\eta_1}$ is monotone.

Proof. From Proposition 2 we know that φ_1 and φ_2 cut at least two times; assume by contradiction that φ_1 and φ_2 cut in three or more points. Under i) or ii), this would imply that $\varphi_1 = \varphi_2$, a contradiction. To prove iii), we note that if φ_1 and φ_2 cut in three or more points, there exist $s_1 < s_2 < s_3 < s_4$ such that $(\varphi_2 - \varphi_1)(s_1) > 0$, $(\varphi_2 - \varphi_1)(s_2) < 0$, $(\varphi_2 - \varphi_1)(s_3) > 0$, $(\varphi_2 - \varphi_1)(s_4) < 0$, or the same inequalities, but with reversed signs. In both cases from the Lagrange theorem the derivative $\varphi_2' - \varphi_1'$ must change sign at least two times; that is, there must exist s_5, s_6 and s_7, with $s_1 < s_5 < s_2 < s_6 < s_3 < s_7 < s_4$, such that $(\varphi_2' - \varphi_1')(s_5) < 0$, $(\varphi_2' - \varphi_1')(s_6) > 0$, $(\varphi_2' - \varphi_1')(s_7) < 0$, or the same inequalities but with reversed signs. Hence:

$$\frac{\varphi_2'(s_5)}{\varphi_1'(s_5)} < 1, \frac{\varphi_2'(s_6)}{\varphi_1'(s_6)} > 1, \frac{\varphi_2'(s_7)}{\varphi_1'(s_7)} < 1,$$

a contradiction with iii). In order to prove iv), we remark that the ratio of elasticities is given by:

$$\frac{\eta_2(s)}{\eta_1(s)} = \frac{\varphi_2'(s)}{\varphi_1'(s)} \frac{\varphi_1(s)}{\varphi_2(s)} = \frac{\frac{d}{ds} \ln \varphi_2(s)}{\frac{d}{ds} \ln \varphi_1(s)}.$$

If φ_1 and φ_2 cut in three points, then also $\ln \varphi_1$ and $\ln \varphi_2$ cut in three points; hence we arrive to a contradiction as in iii). □

We remark that Proposition 4 provides only a sufficient condition for comparability, without specifying which martingale measure is dominating; this will be specified later, in the more general Propositions 6 and 7.

We recall now the classical notion of comparative convexity, that has been introduced by Hardy, Littlewood and Polya and extensively applied in finance by Arrow and by Pratt [28]. For a modern treatment see for example [27].

Definition 3. *Let* $\varphi_1, \varphi_2 : (0, +\infty) \to (0, +\infty)$*. We say that* φ_2 *is more convex than* φ_1 *and write* $\varphi_1 \lhd \varphi_2$ *if* $\varphi_2 = h(\varphi_1)$*, with* h *convex.*

In order to state the properties of the relation $\lhd$, we assume that φ_1 and φ_2 are both strictly increasing or strictly decreasing. From a financial point of view the first case is less realistic, since if $\varphi(s)$ is increasing from the covariance inequality it would follow that:

$$S_0 e^{rT} = E[S_T \varphi(S_T)] \geq E[S_T] E[\varphi(S_T)] = E[S_T],$$

that corresponds to a negative risk premium on the underlying, in contrast with a risk averse representative agent equilibrium.

The following characterizations of comparative convexity are well known:

Proposition 5. *Let* $\varphi_1, \varphi_2 \in C^2(0, +\infty)$ *be both strictly increasing (respectively, decreasing). Then the relationship* $\lhd$ *is a partial preorder. Moreover* $\varphi_1 \lhd \varphi_2$ *is equivalent to any of the following statement:*
a) $\varphi_2 \circ \varphi_1^{-1}$ *is strictly increasing and convex;*
b) $\frac{\varphi_2'}{\varphi_1'}$ *is nondecreasing (resp. nonincreasing);*
c) for each s,

$$\frac{\varphi_2''(s)}{\varphi_2'(s)} \geq \frac{\varphi_1''(s)}{\varphi_1'(s)} \,(\textit{resp. } \frac{\varphi_2''(s)}{\varphi_2'(s)} \leq \frac{\varphi_1''(s)}{\varphi_1'(s)}\,);$$

d) for each random variable X,

$$\varphi_2^{-1} E[\varphi_2(X)] \geq \varphi_1^{-1} E[\varphi_1(X)] \,(\textit{resp. } \varphi_2^{-1} E[\varphi_2(X)] \geq \varphi_1^{-1} E[\varphi_1(X)]\,).$$

Comparative convexity gives a very general sufficient condition for convex comparison of EMMs:

Proposition 6. *Let* $\varphi_1, \varphi_2 \in C^1(0, +\infty)$ *be as in (2), with* $\varphi_1 \neq \varphi_2$*. If* $\varphi_1 \lhd \varphi_2$*, then* $Q_1 \leq_{cx} Q_2$*. More generally, if* $h \circ \varphi_1 \lhd h \circ \varphi_2$ *with* h *strictly increasing, then* $Q_1 \leq_{cx} Q_2$*.*

Proof. If $\varphi_1 \lhd \varphi_2$, then φ_1 and φ_2 cannot cut in three or more points, unless $\varphi_1 = \varphi_2$; it follows that $Q_1 \leq_{cx} Q_2$. Similarly, if $h \circ \varphi_1 \lhd h \circ \varphi_2$ then $h \circ \varphi_2$ and $h \circ \varphi_1$ cannot cut in three or more points, that implies that also φ_1 and φ_2 cannot cut in three or more points. □

Combining this general sufficient condition with the characterizations of comparative convexity, we get the following generalization of Proposition 4:

Proposition 7. *Let* $\varphi_1, \varphi_2 \in C^2(0, +\infty)$ *be as in (2) and decreasing, with* $\varphi_1 \neq \varphi_2$*. If any of the following conditions hold, then* $Q_1 \leq_{cx} Q_2$*:*
i) $\frac{\varphi_2'}{\varphi_1'}$ *is nonincreasing;*
ii) for each s, $\frac{\varphi_2''(s)}{\varphi_2'(s)} \leq \frac{\varphi_1''(s)}{\varphi_1'(s)}$*;*
iii) the ratio of elasticities $\frac{\eta_2}{\eta_1}$ *is nonincreasing;*
iv) for any strictly increasing h, $\frac{\varphi_2'(s)}{\varphi_1'(s)} \frac{h'(\varphi_2)}{h'(\varphi_1)}$ *is nonincreasing.*

We present now some examples. The minimal entropy martingale measure (MEMP henceforth) has a density of the form:

$$\varphi_1(S_T) = \frac{\exp(\lambda_1 S_T)}{E[\exp(\lambda_1 S_T)]},$$

and is obtained through to the minimization of relative entropy defined as

$$H(Q, P) = E_P[\varphi \ln \varphi],$$

that corresponds to the divergence $f(x) = x \ln x$ (see for example [12]).
The Esscher martingale measure, has instead a density of the form:

$$\varphi_2(X_T) = \frac{\exp(\lambda_2 X_T)}{E[\exp(\lambda_2 X_T)]} = \frac{S_T^{\lambda_2}}{E[S_T^{\lambda_2}]},$$

where $X_T = \ln\left(\frac{S_T}{S_0}\right)$ represents the logreturn (see for example [13]).

In [4] we applied the preceding comparison results to show that the MEMP and the Esscher measure are always comparable in the convex order, with the Esscher measure dominating if $E[S_T] > S_0 e^{rT}$, and the MEMP dominating in the opposite case. Indeed, since

$$\eta_1(S_T) = -\lambda_1 S_T, \eta_2(S_T) = -\lambda_2,$$

and since λ_1 and λ_2 have the same sign, the elasticity ratio

$$\frac{\eta_2(S_T)}{\eta_1(S_T)} = \frac{\lambda_2}{\lambda_1 S_T}$$

is decreasing. When $E[S_T] > S_0 e^{rT}$ (positive risk premium on the underlying), both densities are decreasing and Esscher dominates; in the other case $E[S_T] < S_0 e^{rT}$ the densities are increasing and MEMP dominates; a numerical illustration is reported in Fig. 1.

Monfort and Pegoraro [25] introduced second order Esscher measures with the following densities:

$$\varphi(X_T) = \frac{\exp(\lambda X_T + \gamma X_T^2)}{E[\exp(\lambda X_T + \gamma X_T^2)]}, \tag{4}$$

for suitable parameters λ and γ that satisfy the risk neutrality conditions (2). This family has an additional parameter γ so it can be used in econometric estimations of the state price density φ based on real option prices. We show that second order Esscher measures are always ordered; a straightforward computation gives:

$$\eta(S_T) = -\lambda - 2\gamma \ln\left(\frac{S_T}{S_0}\right) = -\lambda - 2\gamma X_T$$

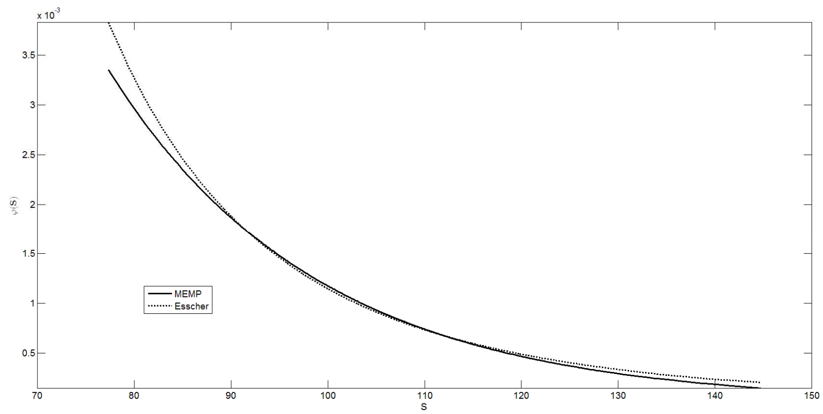

Fig. 1. Convex comparison between Esscher martingale measure (dotted line) and MEMP (continuous line). Since in this example the risk premium on the underlying is positive, the Esscher measure is dominating

and considering different second order Esscher measures Q_1 and Q_2 with parameters respectively λ_1,γ_1 and λ_2,γ_2 the elasticity ratio is given by:

$$r(S_T) = \frac{\eta_1(S_T)}{\eta_2(S_T)} = \frac{\lambda_1 + 2\gamma_1 \ln\left(\frac{S_T}{S_0}\right)}{\lambda_2 + 2\gamma_2 \ln\left(\frac{S_T}{S_0}\right)},$$

that is always monotone since:

$$r'(S_T) = 2\frac{\gamma_1\lambda_2 - \gamma_2\lambda_1}{S_T(\lambda_2 + 2\gamma_2 \ln S_T)^2}.$$

These densities are not necessarily monotone functions of the underlying price at maturity S_T; an example of comparison is reported in Fig. 2.

3 Minimal f-divergences measures

The notion of f-divergence between probability measures has been introduced by Csiszar [8] and independently by Ali and Silvey [1]. We recall the basic definition and properties.

Definition 4. *Let $f : (0, +\infty) \to R$ be convex with $f(1) = 0$. Let Q and P be probability measures on (Ω, F), with $Q \ll P$ and $\varphi = \frac{dQ}{dP}$. The f-divergence $H_f(Q, P)$ between Q and P is defined as follows:*

$$H_f(Q, P) := E_P\left[f\left(\frac{dQ}{dP}\right)\right] = E_P[f(\varphi)]. \quad (5)$$

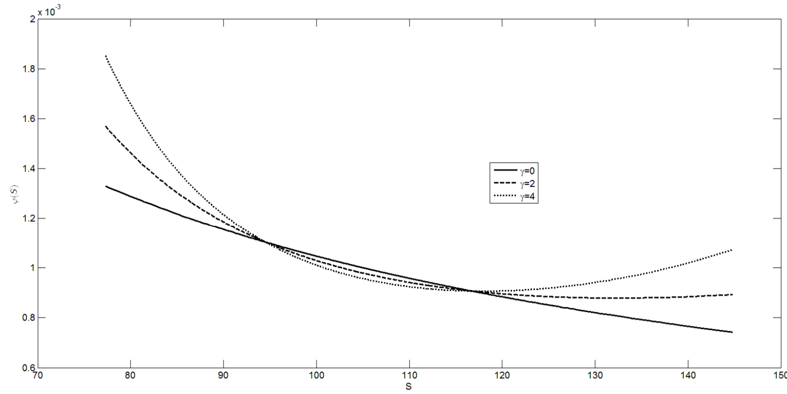

Fig. 2. Convex comparison between second order Esscher martingale measures, for different values of the parameter γ in (4): γ=0 (continuous line), γ=2 (dashed line), γ=4 (dotted line)

The basic properties of f-divergences are collected in the following proposition (see for example the monography [23] and the references therein):

Proposition 8. *Let $H_f(Q,P)$ as in (5). Then we have the following:*
i) $H_f(Q,P) \geq 0$, and if f is strictly convex then $H_f(Q,P) = 0 \Rightarrow Q = P$;
ii) $H_f(Q,P)$ is jointly convex in Q and P and convex in dQ/dP;
iii) If $G \subset F$, then $H_f(Q|_G, P|_G) \leq H_f(Q,P)$;
iv) $H_f(Q,P) = \sup_{G \subset F} H_f(Q|_G, P|_G)$, with G finite partition;
v) $H_f(P,Q) = H_{f^*}(Q,P)$, where $f^*(t) := tf(\frac{1}{t})$.

In the financial literature, Goll and Rüschendorf [15] provided necessary and sufficient conditions for the existence of MDMMs in general semimartingale models. We limit ourselves to the class of the so called power divergences, defined by means of the parametric family f_α (see Fig. 3) given by:

$$f_\alpha(t) = \begin{cases} t\ln t - t + 1 & \alpha = 1 \\ -\ln t + t - 1 & \alpha = 0 \\ \frac{t^\alpha - \alpha(t-1) - 1}{\alpha(a-1)} & \alpha \neq 0,\, \alpha \neq 1 \end{cases}$$

It is elementary to check that $f''_\alpha(t) = t^{\alpha-2}$, $f_\alpha(1) = 0$, $f'_\alpha(1) = 0$; moreover $f_\alpha(t)$ is continuous in α, and $f^*_\alpha(t) = f_{1-\alpha}(t)$. The corresponding f-divergences are called power divergences or also common divergences:

$$H_\alpha(Q,P) := E_P\left[f_\alpha\left(\frac{dQ}{dP}\right)\right],$$

and satisfy $H_\alpha(P,Q) = H_{1-\alpha}(Q,P)$. Many common statistical (pseudo)distances belong to this class: for example H_1 is the relative entropy, H_2 the chi square

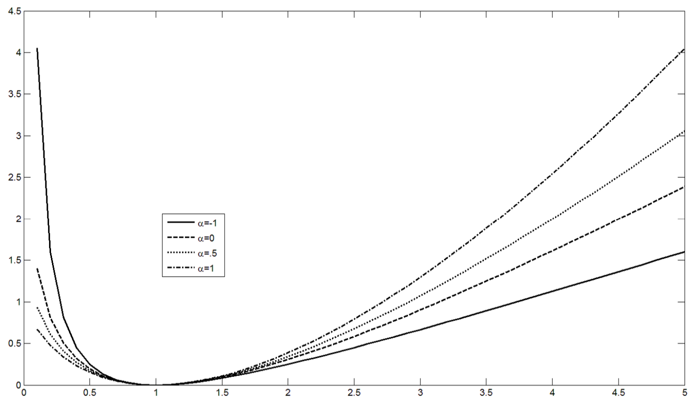

Fig. 3. Power divergences f_α, with $\alpha = -1$ (continuous line), 0 (dashed line), 0.5 (dotted line), 1 (dash-dotted line)

distance, H_0 the reversed relative entropy and $H_{\frac{1}{2}}$ the Hellinger distance, that satisfies the symmetry property $H_{\frac{1}{2}}(Q,P) = H_{\frac{1}{2}}(P,Q)$.
Minimal power divergence martingale measures are defined as follows:

$$Q_\alpha = \arg\min_{Q \in \mathcal{M}} H_\alpha(Q,P).$$

In the context of incomplete markets, minimal power divergences have been considered in many papers; Jeanblanc et al. [21] considered the case of exponential Lévy processes, with $\alpha < 0$ or $\alpha > 1$; Henderson [18] and Henderson et al. [19] considered the case of stochastic volatility models; Cawston and Vostrikova [7] proved that in exponential Lévy models, among all minimal divergence measures, power divergences are the only that preserve the Lévy structure.
We now prove that in the one period setting minimal power divergence martingale measures are ordered. We have the following

Theorem 1. *Let Q_α and Q_β as in the preceding definition, with $\alpha \leq \beta$; then $Q_\beta \leq_{cx} Q_\alpha$. This implies that option prices are decreasing in the parameter α.*

Proof. The problem is

$$\begin{cases} \min_{\varphi} E_P[f_\alpha(\varphi)] \\ E_P[\varphi S_T] = S_0 e^{rT} \\ E_P[\varphi] = 1 \end{cases}$$

The Lagrangian is given by:

$$L(\varphi, \lambda, \mu) = E_P[f_\alpha(\varphi)] - \lambda(E_P[\varphi S_T] - S_0 e^{rT}) - \mu(E_P[\varphi] - 1),$$

and the first order conditions developed in [15] become simply:

$$\begin{cases} f'_\alpha(\varphi^\star_\alpha) = \lambda S_T + \mu \\ E_P[\varphi^\star_\alpha S_T] = S_0 e^{rT} \\ E_P[\varphi^\star_\alpha] = 1 \end{cases}$$

Since in the power case

$$f'_\alpha(t) = \begin{cases} \ln t & \alpha = 1 \\ \frac{t^{\alpha-1}-1}{a-1} & \alpha \neq 1 \end{cases}$$

the densities of the minimal divergence measures are of the form:

$$\varphi^\star_\alpha(S_T) = \begin{cases} \exp(\lambda S_T + \mu) & \alpha = 1 \\ [1 + (\alpha-1)(\lambda S_T + \mu)]^{\frac{1}{\alpha-1}} & \alpha \neq 1 \end{cases} \tag{6}$$

where the parameters λ and μ are chosen in order to satisfy the constraints $E_P[\varphi^\star_\alpha] = 1$ and $E_P[\varphi^\star_\alpha S_T] = S_0 e^{rT}$. The elasticity is given by:

$$\eta^\star_\alpha(S_T) = \frac{\lambda S_T}{1 + (\alpha-1)(\lambda S_T + \mu)}.$$

The elasticity ratio between two different minimal divergence martingale measures Q_α and Q_β is given by:

$$r_{\alpha,\beta}(S_T) = \frac{\eta^\star_\alpha(S_T)}{\eta^\star_\beta(S_T)} = \frac{\lambda_\alpha}{\lambda_\beta} \frac{1 + (\beta-1)(\lambda_\beta S_T + \mu_\beta)}{1 + (\alpha-1)(\lambda_\alpha S_T + \mu_\alpha)},$$

that is always a monotone function of S_T; from Proposition 4, item iv), it follows that Q_α and Q_β are comparable in the convex order. Since from (6) it is easy to see that the lower α, the more convex is $\varphi^\star_\alpha(S)$, from Proposition 6 we have the thesis. □

A numerical illustration is given in Fig. 4 and Fig. 5.

4 Multiperiod extension

The computation of MDMMs in discrete time models has been considered by several authors: Gzyl [17] and Ssebugenyi [31] considered the case of the MEMP, while Grandits [16] and Arai and Kawaguchi [2] considered p-optimal martingale measures. We begin by introducing some standard notations. Let (Ω, F, F_n, P) be a filtered probability space with $F = \{\emptyset, \Omega\}$ and $F_N = F$. The dynamics of the riskless asset is $B_k = B_{k-1} e^{r_k}$, with $r_k \in F_{k-1}$; the dynamics of the risky asset is $S_k = e^{X_k} S_{k-1}$, with $X_k \in F_k$. The no arbitrage condition corresponds to the requirement that

$$P(X_k > r_k | F_{k-1}) > 0 \text{ and } P(X_k < r_k | F_{k-1}) > 0, \text{ for } k = 1, ..., N,$$

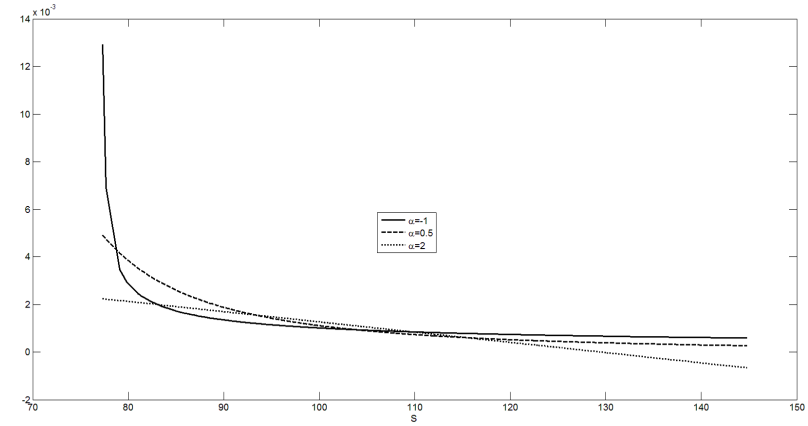

Fig. 4. Convex comparison between minimal power divergence martingale measures with α=-1 (continuous line), α=0.5 (dashed line), $\alpha = 2$ (dotted line). The convex order is reversed with respect to the parameter α

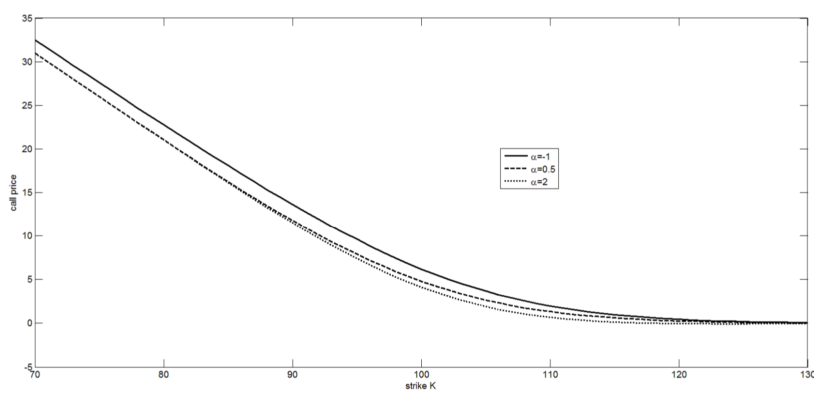

Fig. 5. Comparison between call prices given by minimal power divergence martingale measures α=-1 (continuous line), α=0.5 (dashed line), $\alpha = 2$ (dotted line). Option prices are decreasing with respect to the parameter α

and is equivalent to the existence of an EMM for the discounted process.
The density process of any EMM is defined as

$$Z_n := E_P\left[\frac{dQ}{dP}|F_n\right],$$

and can always be written as a product of *one period* change of measures φ_k; that is, we can always write:

$$Z_n = \prod_{k=1}^{n} \varphi_k, \text{ with } \varphi_k = \frac{Z_k}{Z_{k-1}} > 0 \text{ a.s. and } E_P[\varphi_k|F_{k-1}] = 1. \tag{7}$$

We have that Q is a martingale measure for the discounted price process if and only if:

$$E_P[\varphi_k e^{X_k}|F_{k-1}] = e^{r_k}, \text{ for } k = 1, ..., N$$

(see for example [30] for a general treatment of multiperiod discrete time models).

The computation of MDMMs in multiperiod models is not trivial and typically requires backward induction procedures. In particular, in general the density of the MDMM cannot be written as the product of one period divergence-minimizing densities as in (7).

We illustrate this point by means of a simple two period trinomial example. The price process is an additive random walk with dependent increments:

$$S_1 - S_0 = \begin{cases} 1 & 1/2 \\ 0 & 3/8 \\ -1 & 1/8 \end{cases}, \quad S_2 - S_1 = \begin{cases} \widetilde{U} & \text{if } S_1 = 1 \\ U & \text{if } S_1 = -1 \text{ or } 0 \end{cases} \tag{8}$$

$$\text{with } U = \begin{cases} 1 & 1/3 \\ 0 & 1/3 \\ -1 & 1/3 \end{cases} \text{ and } \widetilde{U} \stackrel{d}{=} S_1 - S_0$$

the corresponding tree is reported in Fig. 6. It is not difficult to check that the distribution of the underlying under the minimal entropy martingale measure is the following:

$$S_1 - S_0 = \begin{cases} 1 & q_1 \\ 0 & 1-2q_1 \\ -1 & q_1 \end{cases}, \quad S_2 - S_1 = \begin{cases} \stackrel{d}{=} \overline{U} & \text{if } S_1 = 1 \\ \stackrel{d}{=} U & \text{if } S_1 = -1 \text{ or } 0 \end{cases}$$

$$\text{with } \overline{U} = \begin{cases} 1 & q_2 \\ 0 & 1-2q_2 \\ -1 & q_2 \end{cases}, \text{ with } q_1 \simeq 0.2775 \text{ and } q_2 \simeq 0.2857.$$

Clearly, since $q_1 \neq q_2$, this is not the product of one period entropy-minimizing densities.

On the other hand, as was suggested in [17] and [14], it is very natural to consider *local* MDMMs, defined *by construcion* as the product of one period

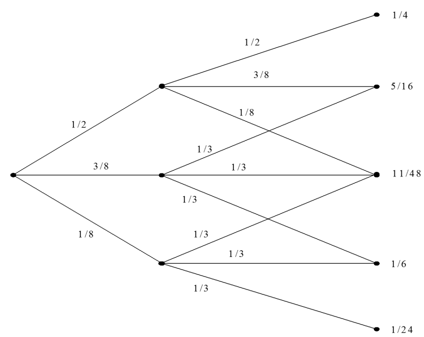

Fig. 6. A two period trinomial example

divergence-minimizing densities. That is, a local MDMM is defined by means of the density process:

$$\widetilde{Z}_n := \prod_{k=1}^{n} \widetilde{\varphi}_k, \tag{9}$$

where each $\widetilde{\varphi}_k$ is the solution of the problem:

$$\begin{cases} \min_{\varphi} E_P[f(\varphi_k)|F_{k-1}] \\ E_P[\varphi_k|F_{k-1}] = 1 \\ E_P[\varphi_k e^{X_k}|F_{k-1}] = e^{r_k} \end{cases} \tag{10}$$

In the preceding example the local MEMP would be given by:

$$S_1 - S_0 = \begin{cases} 1 & q \\ 0 & 1-2q \\ -1 & q \end{cases}, \qquad S_2 - S_1 = \begin{cases} \overset{d}{=} \overline{U} & \text{if } S_1 = 1 \\ \overset{d}{=} U & \text{if } S_1 = -1 \text{ or } 0 \end{cases}$$

$$\text{with } \overline{U} = \begin{cases} 1 & q \\ 0 & 1-2q_2 \\ -1 & q \end{cases}, \quad \text{with } q \simeq 0,2857.$$

In [4] it was shown that convex comparison between one period changes of measure implies the convex comparison of their product; more precisely, the following Lemma was proved:

Lemma 1. *Let* $\varphi^1 = \prod_{k=1}^{n} \varphi_k^1$ *and* $\varphi^2 = \prod_{k=1}^{n} \varphi_k^2$*, with* φ_k^i *satisfying* $\varphi_k^i > 0$ *a.s.,* $E_P[\varphi_k^i|F_{k-1}] = 1$*,* $E_P[\varphi_k^i e^{X_k}|F_{k-1}] = e^{r_k}$*. If for each* k *we have that* $\varphi_k^1 \leq_{cx} \varphi_k^2$*, then* $\varphi^1 \leq_{cx} \varphi^2$*.*

This Lemma was applied to the convex comparison of the multiperiod Esscher measure (that is *local* by construction) and the local MEMP. Similarly, we can establish convex comparison between local power MDMMs:

Theorem 2. *Let Q_α and Q_β be local power MDMMs, defined by (9) and (10). If $\alpha \leq \beta$, then $Q_\beta \leq_{cx} Q_\alpha$.*

As a final example, we compute local and global power MDMMs for α ranging from 1 (the case of entropy) to 5, in the two period trinomial model (8). The results are reported in Table 1 and show that indeed the local MDMMs are decreasing in the convex order with respect to α, in accordance with Theorem 2; moreover, the lower part of the table suggests that a similar result could hold also for the global MDMMs, but the proof is still a open problem.

Table 1. Convex comparison between *local* (upper part) and *global* (lower part) minimal power divergence martingale measures, for different values of α in the two period trinomial model (8). Both families appear to be ordered in the convex order

Local MDMMs	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
$S_2 = 2$	0.0816	0.0666	0.0570	0.0518	0.0488
$S_2 = 1$	0.2653	0.2862	0.2988	0.3056	0.3093
$S_2 = 0$	0.3197	0.3139	0.3108	0.3092	0.3085
$S_2 = -1$	0.2381	0.2473	0.2537	0.2575	0.2597
$S_2 = -2$	0.0952	0.0860	0.0796	0.0759	0.0737
Global MDMMs	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$	$\alpha = 4$	$\alpha = 5$
$S_2 = 2$	0.0793	0.0648	0.0563	0.0515	0.0487
$S_2 = 1$	0.2673	0.2874	0.2993	0.3057	0.3094
$S_2 = 0$	0.3201	0.3144	0.3110	0.3094	0.3086
$S_2 = -1$	0.2408	0.2497	0.2548	0.2579	0.2598
$S_2 = -2$	0.0925	0.0837	0.0785	0.0754	0.0735

5 Conclusions and directions for further research

We have seen that in the one period case it happens quite often that two different equivalent martingale measures Q_1 and Q_2 are comparable in the convex order. In Propositions 4, 6 and 7 we have presented new sufficient conditions for the convex ordering that generalize those already proved in [11] and [4]. In particular, it seems that minimal power divergence martingale measures are always ordered (Theorem 1). This ordering partially extends to the multiperiod case (Theorem 2), and reflects a similar result already found in the context of stochastic volatility models (see [18], [19]). A natural question is if it is possible to extend these result beyond the class of power divergences; another natural question is if in the multiperiod case is possible to establish comparison result between *global* martingale measures; both issues are the subject of current research.

Finally, I would like to thank three anonymous referees for their corrections and suggestions, that have lead to an improvement of the paper.

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Mathematical Methods in Economics and Finance – m²ef
Vol. 7, No. 1, 2012
ISSN print edition: 1971-6419 – ISSN online edition: 1971-3878
Web page: `http://www.unive.it/m2ef/` – E-mail: `m2ef@unive.it`