

Risk indicators in equity markets*

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Abstract. The distribution of securities prices in financial markets is known to exhibit heavy tails, and furthermore the time trajectory has occasional extreme swings or reversals in direction. The modelling of heavy tails has been achieved with the addition of a homogeneous point process to a diffusive process. However, the timing of the jumps in the point process should capture the price reversals. In this paper a non-homogeneous point process is introduced, so that the intensity and size of jumps are state dependent. The state is characterized by stress measures, which are composed from combinations of risk factors. The factors considered are the bond-stock yield differential and the volatility index. The parameters in the model are estimated from data on the US market from 1990 - 2007. An out-of-sample test is performed for 2008 - 2009. The model captures the swings in equities prices and provides a basis for anticipating reversals from risk factors.

Keywords. Cox process, endogenous instability, stress factors, extreme equity risk, maximum likelihood.

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1 Introduction

The short and medium term dynamics of equity markets in developed economies have in recent years (e.g. last two decades) increasingly shown evidence of fundamental imbalances associated with frequent volatility regime switching and occasional prolonged periods of one-sided tail events. The development of new

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modeling approaches capturing those complex dynamics in several equity markets (e.g. US, Japan, UK) has been unprecedented with a remarkable impact on financial practice and investment decisions (see [6], [20], [23] as recent references). A range of possible modeling frameworks are now available to fit highly skewed and fat tailed as well as canonical market distributions and adapted to risk assessment and portfolio optimization applications. In early studies [1], [14], [4] we have presented an equity return generation model for the US market characterised by a continuous diffusion process with random drift and a Cox process with state-dependent intensity and shocks. The associated stock price process belongs to the class of right-continuous left-limited (RCLL) processes. A key role, in the proposed framework, is played by what we call the risk or instability process defined uniquely by the point process (see below equation (5)). Relative to earlier approaches the model sets a clear and statistically testable relationship between equity market reversals and an underlying source of risk whose behaviour can be inferred from market trends. The introduction of an inner source of instability driving the market volatility provides a clear generalization of previous approaches with a potential to link the economics of financial instability and its statistical characterization.

From a financial viewpoint the possibility to identify stress factors inducing large inflows and outflows in the equity market is of primary importance for strategic allocation decisions and policy makers interventions. The former Federal Reserve Chairman Alan Greenspan has indeed pushed forward, in his famous 1996 irrational exuberance speech, the idea that the bond-stock earning differential should have been considered to assess equity markets relative misvaluation [1], [18] before the 1987 crisis. Ziemba and Schwartz [22] had already considered a similar risk measure in 1991. The economic rationale being that in the very long run the equity market is expected to fluctuate around a theoretical value determined by the market earning expectations and a discount factor reflecting the 10 year interest rate behavior. This measure has been shown in [4] to occasionally underestimate the observable market dynamics over the 1980-2005 period in the US. Furthermore a risk process uniquely driven by the yield differential has been shown to be statistically significant and appropriate to capture market reversals, though neither necessary nor sufficient to anticipate observed market shocks.

We propose in this work an extension of the financial model to account for an additional risk factor directly generated by the options market. The Chicago Board Options Exchange (CBOE) defines an implied volatility measure called the VIX. The index is updated daily on the basis of 30 day ATM options traded on the S&P 500 index. As such the index reflects investors' expectations of forward market movements. According to the structural default model [16], [3], the aggregate implied volatility in the equity market, capturing the relationship between leverage and equity returns, is a key variable in assessing the credit cycle and it is heavily correlated with the prevailing spreads in the credit markets.

If the rationale for the inclusion of the bond yield differential as a potential stress measure comes from the (continuously varying) expected impact of the

risk free interest rate and the market expectation over future earnings, the consideration of the VIX does reflect both an aggregate measure of an economy wide credit cycle (according to the structural approach to credit risk) and a direct signal of the market uncertainty over future corporate performance. An increasing implied volatility indicator would then reflect a generalized contraction of investors planning horizons and a reduction of corporate earnings, resulting into an equity market downturn.

The ultimate aim of our study is twofold: we introduce a general statistical procedure based on a recursive maximum likelihood estimator to characterize underlying risk processes determining equity returns, and we propose the introduction of two risk indicators with the potential to anticipate market reversals. The presented methodology can be generalized to many instability factors, and alternative indicators can be put forward to capture the equity market sentiment. In this article we show that the introduction of market stress measures, which are combinations of risk factors (we just consider two such factors), jointly with a canonical geometric Brownian motion model may in certain periods improve the fitting of market dynamics even during periods of severe instability. The study is conducted over the 1990-2009 period in the US market.

In Section 2 of this paper the risk indicators impacting the securities market are defined. The dynamics for securities prices are separated into a diffusion and a nonhomogeneous point process, referred to as the risk process. The risk is modeled by Weibull processes, with jump size and intensity parameters depending monotonically on the risk factors through stress measures. A conditional maximum likelihood estimation procedure is discussed in Section 3, where the identification of jump times in the point process follows from the monotonicity in the stress measures. In Section 4 the methodology is applied to market data on stocks and bonds in the US for the period 1990 - 2009. The market implications of the pricing model and in particular the risk process are discussed in Section 5.

2 Risk indicators

The focus of our study is represented by the equity market (here specifically the US equity market). To account for investment movements from and towards the fixed income market we present a stochastic model for the equity and the bond market.

Consider in particular: $S(t)$ = stock price at time t ; $B(t)$ = bond price at time t . The stock and bond prices are random variables defined on a probability space (Ω, \mathcal{F}, P) , representing the uncertain dynamics of the market. With the prices on a log scale, let $Y_1(t) = \ln(B(t))$, $Y_2(t) = \ln(S(t))$.

It is assumed that the dynamics of price movements are defined by geometric Brownian motion for bonds, and geometric Brownian motion plus a marked point process for stocks. The level I or conditional log-price dynamics, given parameter values and initial conditions $Y_1(0) = y_1$, $Y_2(0) = y_2$, are defined by the equations

$$dY_1(t) = [\mu_1(t)dt + \delta_1(t)dW_1(t)] \quad (1)$$

$$dY_2(t) = [\mu_2(t)dt + \delta_2(t)dW_2(t)] + [dR(t)]. \quad (2)$$

In these equations, W_{1t} and W_{2t} are independent standard Wiener processes. It is assumed that the drift parameters $\alpha_1(t)$ and $\alpha_2(t)$ are random variables, whose distributions are affected by common market forces. Those common forces also generate the correlation structure between the prices on stocks and bonds. For the drift, the dependence on factors is implicit rather than explicit. Consider the market factor, given by $F(t)$ at time t , where $F(t)$ is a standard Gaussian variable. Then the random drift parameters are

$$\mu_1(t) = \mu_1 + \gamma_1 F(t) \quad (3)$$

$$\mu_2(t) = \mu_2 + \gamma_2 F(t). \quad (4)$$

These equations represent the effect of the market forces on the direction of asset prices. The log-prices for the assets are correlated, with the correlation captured by the relationship to the common factor F , as defined by (γ_1, γ_2) . Therefore $\text{corr}(dY_1, dY_2) = \gamma_1 \gamma_2$.

The idiosyncratic volatility parameters are assumed to be deterministic, so $\delta_1(t) = \delta_1$, and $\delta_2(t) = \delta_2$. The full set of parameters in the Brownian motion component are represented as $\Theta = (\mu_1, \mu_2, \gamma_1, \gamma_2, \delta_1, \delta_2)$.

The risk component $dR(t) = dR_1(t) + dR_2(t)$, where $dR_1(t)$ and $dR_2(t)$ represent up and down shocks respectively, is a marked point process with time/state dependent sizes and intensities. The separate point processes determine the shocks to the stock price. The components of the risk processes are assumed to be affected by multiple risk factors, $X = (X_1, \dots, X_J)$. So the size $\vartheta_i(X)$ and the intensity $\lambda_i(X)$ of up, $i = 1$, and down, $i = 2$, shocks depend on X . It is assumed that up and down shocks are mutually exclusive. The rationale for including separate processes for up and down shocks is the possible differences in investor reactions to high and low values of the risk factors.

The risk process dynamics are

$$dR(t) = \vartheta_1(t)dN_1(\lambda_1(t)) + \vartheta_2(t)dN_2(\lambda_2(t)). \quad (5)$$

The processes N_1 and N_2 characterize up shocks ($E(\vartheta_1(t)) > 0$) and down shocks ($E(\vartheta_2(t)) < 0$), respectively. There are two factors or risk indicators which are considered in this analysis: (i) the differential in yields on stocks and bonds; (ii) the implied volatility of stocks. To capture the stress from yields, consider the variables: $U(t)$ = the stock market implied yield at time t ; $r(t)$ = bond market yield at time t . Let $\nu(t) = \frac{r(t)}{U(t)}$ = ratio of bond yield to stock yield, and ν^* = the average or long term yield ratio. Then the following yield variables are proposed as risk factors.

1. *Yield up*

$$X_{11}(t) = \max \left\{ \frac{\nu^*}{\nu(t)}, 1 \right\}. \quad (6)$$

This factor is a force for an upward shock in equity prices based on the undervaluation of securities.

2. *Yield down*

$$X_{12}(t) = \max \left\{ \frac{\nu(t)}{\nu^*}, 1 \right\}. \quad (7)$$

This factor is a force for a down shock in equity prices based on the overvaluation of securities.

The Chicago Board Options Exchange (CBOE) defines an implied volatility measure called the VIX. It represents the expected forward volatility of the equity index around the risk free rate over the upcoming 30 day period (on an annualized basis). The movement can be up or down. The VIX is reported as a percentage. Since its introduction the indicator varied from historically lows of 10% up to a maximum level of 50%. A large VIX value indicates an expectation of a sharp increase of equities price volatility or shocks in the terminology of this work. The risk factor based on implied volatility is defined by the VIX:

$$X_2(t) = 1 + \frac{VIX}{100}. \quad (8)$$

The direction of a shock is not revealed by the VIX, so another indicator is required. The obvious direction indicator is the yield ratio.

3. *Volatility: Up*

$$X_{21}(t) = \begin{cases} X_2(t) & \text{if } \nu(t) < \nu^* \\ 1 & \text{if } \nu(t) \geq \nu^* \end{cases}. \quad (9)$$

4. *Volatility: Down*

$$X_{22}(t) = \begin{cases} X_2(t) & \text{if } \nu(t) < \nu^* \\ 1 & \text{if } \nu(t) \geq \nu^* \end{cases}. \quad (10)$$

It is anticipated that the risk factors impact investor decisions. If equities are overvalued, as determined by the yield ratio, then downward pressure on stock prices grows and the likelihood of a shock increases. An expectation of volatility in the equities market, as measured by the VIX, is an indication of an impending shock. It is possible that the risk factors act separately or in combination. The mechanism is stress or market discordance measures, with measure k defined as

$$\pi_{ik}(t) = \prod_{j=1}^J X_{ji}^{w_{jk}}(t), \quad (11)$$

where $w_{jk} > 0$, $\sum_{j=1}^J w_{jk} = 1$. On the log scale

$$\psi_{ik}(t) = \ln(\pi_{ik}(t)) = \sum_{j=1}^J w_{jk} \cdot \ln(X_{ji}).$$

The stress measures are analogous to principal components generated from the risk factors. Since the measures are constructed to explain price movements

rather than correlation between risk factors, the linear combinations will differ from components.

The theory for the shock processes in this paper proposes that the shock intensities depend monotonically on the stresses generated by the risk factors, with increased stress implying a greater chance of a shock. Let $\pi_{ik}, i = 1, 2, k = 1, \dots, K$ be the stress for an up and down shocks respectively. An increasing intensity implies a Weibull process, so that π_{ik} follows a Weibull distribution with density for $i = 1, 2$

$$f_{ik}(\pi_{ik}) = \frac{\beta_{ik}}{\phi_{ik}} \left(\frac{\pi_{ik}}{\phi_{ik}} \right)^{\beta_{ik}-1} e^{-\left(\frac{\pi_{ik}}{\phi_{ik}}\right)^{\beta_{ik}}}. \quad (12)$$

The cumulative distribution is $F_{ik}(\pi_{ik}) = 1 - e^{-\left(\frac{\pi_{ik}}{\phi_{ik}}\right)^{\beta_{ik}}}$. The cumulative can be written in terms of the stress measure as

$$\ln\{-\ln[1 - F_{ik}(\pi_{ik})]\} = -\beta_{ik} \ln(\phi_{ik}) + \beta_{ik} \psi_{ik}(t). \quad (13)$$

The intensity associated with stress measure k is

$$\lambda_{ik}(t) = \frac{f_{ik}(\pi_{ik})}{1 - F_{ik}(\pi_{ik})}. \quad (14)$$

With the Weibull processes, where π has a Weibull distribution, it is known that $\psi_{ik} = \ln(\pi_{ik})$ has an extreme value distribution. To have the shock size reflecting extreme returns, it is assumed that size depends linearly on $\psi_{ik}(t)$. If there is a shock at time t , the size is assumed to be

$$\vartheta_{ik}(t) = \theta_{ik0} + \theta_{ik1} \psi_{ik}(t) + \eta_{ik} Z_{ik}(t) \quad (15)$$

where $Z_{ik}(t)$, are independent, standard Gaussian variables. So the expected shock size is proportional to the stress. With the stress related intensity and size defined, the risk process dynamics are

$$R(t) = \sum_k (\vartheta_{1k}(t) dN(\lambda_{1k}(t)) + \vartheta_{2k}(t) dN(\lambda_{2k}(t))). \quad (16)$$

The distinguishing feature of the asset pricing model is the risk process. The parameters in the risk process for a stress measure are $\Xi = (\Xi(1), \dots, \Xi(J))$, where $\Xi(k) = (\theta_{1k0}, \theta_{1k1}, \theta_{2k0}, \theta_{2k1}, \eta_{1k}, \eta_{2k}, \phi_{1k}, \phi_{2k}, \beta_{1k}, \beta_{2k})$. It is hypothesized that the risk factors characterize market stress, which in turn affects shocks to equity prices through the model parameters: $X \rightarrow \pi \rightarrow \Xi \rightarrow R$. In subsequent sections these relationships will be explored with price data on bonds and stocks in the US financial market.

3 Parameter estimation

The methods in this section provide estimates for the parameters (Θ, Ξ) in the model for asset price dynamics. The parameters in Ξ depend on the risk factors,

and define the risk process. So a framework is in place to study the risk process and the risk factors.

Consider the set of observations at regular intervals in time (days) of log-prices for bonds and stocks

$$\{y_1, \dots, y_t\},$$

where

$$y'_s = (y_{1s}, y_{2s}), s = 1, \dots, t.$$

Let the observed daily changes in log-prices be

$$e_s = y_s - y_{s-1}, s = 1, \dots, t. \quad (17)$$

The risk factors are needed for the period preceding a shock. The preferred measure of stock yield is the earnings - price ratio. For the yields on stocks, the earnings-price ratio is recorded: $\hat{U}_s = \frac{E_s}{S_s}$. With the rate on the 10-year bond r_s , the yield ratio is $\hat{v}_s = \frac{r_s}{\hat{U}_s}$. The VIX values are reported by the CBOE.

It is assumed there is an unobservable set of random shocks imbedded in the observed increments. Let $e = (e_1, \dots, e_t)$ be the vector of observations and

$$I = (I_{11}, \dots, I_{1t}, I_{21}, \dots, I_{2t})$$

be the associated shock indicators, where $I_{is} = 1$ for a shock, and $I_{is} = 0$ otherwise and $I_{1s} \cdot I_{2s} = 0, s = 1, \dots, t$. The conditional likelihood is $L(\Theta, \Xi) = p(e, I) = p(e|I) \times p(I)$, and then, the log of the likelihood is $l(\Theta, \Xi) = \ln(p(e, I)) = \ln(p(e|I)) + \ln(p(I))$. The diffusion (random walk) parameters and the jump size parameters can be estimated using the conditional log likelihood $l(\Theta|I^*) = \ln(p(e|I^*))$ for a given jump sequence I^* . Note that $\ln(p(e|I)) = \ln \prod_{s=1}^t p(e_s|I_s) = \sum_{s=1}^t \ln(p(e_{is}|I_s))$. The intensity parameters (ϕ_{1k}, β_{1k}) and (ϕ_{2k}, β_{2k}) can be estimated independently from $\ln(p(I))$ using the Weibull distribution.

If the shock sequences can be identified, conditional maximum likelihood can be used to estimate parameters. The hypothesis in the model is that the risk process parameters depend *monotonically* on the risk factors through the market stress functions. If there are thresholds such that the chance of a shock is almost certain, then identifying the thresholds is critical. The approach is to set threshold values for extreme stress, and times where the value is exceeded are identified. If threshold values for high stress are (π_{1k}^*, π_{2k}^*) , then the increments with $\{\pi_{1k} > \pi_{1k}^*\}$ identify up shock times and $\{\pi_{2k} > \pi_{2k}^*\}$ identify down shock times. Given those shock times, conditional maximum likelihood estimates for model parameters are determined. The value of the parameter estimates and the conditional likelihood are compared for combinations of $\{(\pi_{1k}^*, \pi_{2k}^*)\}$, and appropriate thresholds for up/down shocks are established. Since the expectation is that the risk/shock component of a price movement dominates the diffusion, the positive and negative price changes will be used as a secondary indicator of a shock period.

In considering market stress and the risk factors, there are two issues. (i) the stress thresholds (π_{1k}^*, π_{2k}^*) ; (ii) the weights assigned to the risk factors

$\{w_{jk}, j = 1, \dots, J\}$. It is important to identify thresholds where shocks are likely to occur, but it is also useful to identify the relative importance of the factors. The method proceeds as follows:

1. Set the weights $\{w_{jk}, j = 1, \dots, J\}$, and calculate the stress values $\pi_{ik}(t), i = 1, 2$.
2. Calculate the empirical distribution \hat{F}_{1k} for stress values π_{1k} , and \hat{F}_{2k} for stress values π_{2k} over the study period $(1, t)$.
3. Specify a grid size $\omega > 0$, an initial tail area p for deviations, and a step number m_{\max} . Set $m = 0$.
4. With a grid point m and a tail area $\alpha_m = p - m\omega$, identify times/indices

$$T_{1m} = \left\{ s \mid [e_s \geq 0] \wedge [\hat{F}_1(\pi_{1,s-1}) \geq 1 - \alpha_k] \right\}$$

$$T_{2m} = \left\{ s \mid [e_s < 0] \wedge [\hat{F}_2(\pi_{2,s-1}) \geq 1 - \alpha_k] \right\}.$$

5. Assume there is an up shock at times $s \in T_{1m}$, and a down shock for times $s \in T_{2m}$. For this sequence of shocks, calculate the conditional maximum likelihood estimates for model parameters.
6. Reset m and return to [3].
7. Select the thresholds and shock times which provide the best fit - minimum mean squared error.
8. Reset the weights and repeat the process.

For given thresholds and the identified shock times, a conditional likelihood maximization is required. The diffusion and the jump size parameters are estimated by maximizing the log-likelihood, $l(\Theta|I) = \ln(p(e|I))$ for a given sequence I . The Weibull parameters are estimated from $l(\Xi|I) = \ln(p(I))$.

The conditional distribution for e_s given I and Θ is a bivariate normal distribution with mean vector $\xi_s(I, \Theta)$ and covariance matrix, $\Sigma_s(I, \Theta)$, respectively,

$$\begin{pmatrix} \xi_{1s} \\ \xi_{2s} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 + I_{1s} \sum_k (\theta_{1k0} + \psi_{1ks} \theta_{1k1}) + I_{2s} \sum_k (\theta_{2k0} + \psi_{2ks} \theta_{2k1}) \end{pmatrix} \quad (18)$$

$$\begin{pmatrix} \sigma_{1s}^2 & \sigma_{12} \\ \sigma_{12} & \sigma_{2s}^2 \end{pmatrix} = \begin{pmatrix} \gamma_1^2 + \delta_1^2 & \gamma_1 \gamma_2 \\ \gamma_1 \gamma_2 & \gamma_2^2 + \delta_2^2 + \sum_k \eta_{1k}^2 I_{1s} + \sum_k \eta_{2k}^2 I_{2s} \end{pmatrix}. \quad (19)$$

It is informative to note that the mean and volatility for stocks are time varying. For the given values of the shocks indicators $I = ((I_{11}, \dots, I_{1t}), (I_{21}, \dots, I_{2t}))$, yield deviations $((\hat{\psi}_{1,0}, \dots, \hat{\psi}_{1,t-1}), (\hat{\psi}_{2,0}, \dots, \hat{\psi}_{2,t-1}))$ and data $e = (e_1, \dots, e_t)$, the data can be split into sets based on times with shocks. Let $A_i = \{t|I_{it} = 1\}$, $i = 1, 2$, and $\bar{A} = \{s|I_{is} = 0, i = 1, 2\}$.

Consider the statistics on increments e for the subsamples: (i) the number of values n_{A_i} , $i = 1, 2$, $n_{\bar{A}}$; (ii) means - $\hat{\xi}_{A_i}$, $i = 1, 2$, $\hat{\xi}_{\bar{A}}$; (iii) covariance matrices S_{A_i} , $i = 1, 2$, $S_{\bar{A}}$. The subsample statistics are the basis of maximum likelihood

estimates for parameters. The conditional likelihood for the diffusion and shock size parameters is

$$L(\Theta, \Xi|I, \hat{\psi}, e) = (2\pi)^{-t} \prod_{s=1}^t |\Sigma_s(I)|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} \sum_{s=1}^t (e_s - \xi_s(I))' \Sigma_s^{-1} (e_s - \xi_s(I)) \right\}. \quad (20)$$

The function $l(\Theta, \Xi|I, \hat{\psi}, e) = \ln(L(\Theta, \Xi|I, \hat{\psi}, e))$ is solved iteratively for conditional maximum likelihood estimates. The structure in the covariance matrix is important for model fitting. Consider $\Gamma' = (\gamma_1, \gamma_2)$ and

$$\Delta_s(I_s) = \begin{bmatrix} \delta_1^2 & 0 \\ 0 & \delta_2^2 + \sum \eta_{1k}^2 I_{1s} + \sum \eta_{2k}^2 I_{2s} \end{bmatrix}.$$

Then $\Sigma_s(I) = \Gamma\Gamma' + \Delta_s(I_s)$. This decomposition of the covariance matrix into a matrix determined by common market factors and a matrix of specific variances will be important in the estimation of parameters. For a given covariance matrix, $\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$, it is possible to write the solution to the structural equation: $\Gamma' = (\sqrt{\rho}\sigma_1, \sqrt{\rho}\sigma_2)$, $\Delta = \text{diag}((1 - |\rho|)\sigma_1^2, (1 - |\rho|)\sigma_2^2)$, where $\rho = \frac{\sigma_{12}}{\sigma_1\sigma_2}$. So estimates for the covariance naturally lead to estimates for the parameters Γ and Δ .

With a given sequence of shocks, we can estimate $(\phi_{1k}, \beta_{1k}, \phi_{2k}, \beta_{2k})$ using the Weibull distribution. The power law intensity (Weibull) implies that the stress measure is the driving force in the occurrence of a shock.

Consider the data on stress measures π_i at the actual shock times I ,

$$\mathbf{x}_i = \{x_{i1}, \dots, x_{in_{A_i}}\}, \quad i = 1, 2.$$

With this data, consider the likelihood equations

$$G(\beta_{ik}|\mathbf{x}_i) = n_{A_i} \sum_{j=1}^{n_{A_i}} x_{ij}^{\beta_{ik}} + \left(\sum_{j=1}^{n_{A_i}} \ln(x_{ij}) \right) \sum_{j=1}^{n_{A_i}} \beta_{ik} x_{ij}^{\beta_{ik}} - n_{A_i} \sum_{j=1}^{n_{A_i}} \beta_{ik}^2 x_{ij}^{\beta_{ik}-1}, \quad i = 1, 2.$$

The conditional maximum likelihood estimates for the Weibull parameters (ϕ_i, β_i) are $(\hat{\phi}_i, \hat{\beta}_i)$, where $G(\hat{\beta}_{ik}|x_i) = 0$ and $\hat{\phi}_{ik} = \frac{\sum_{j=1}^{n_{A_i}} x_{ij}^{\hat{\beta}_{ik}}}{n_{A_i}}, i = 1, 2.$

All estimation routines were programmed in Matlab.

4 Market fitting

Data on the stock and bond markets in the United States are now analyzed using the proposed model for asset prices. The objectives are to determine: (i) if the risk process improves the fitting of a model to an actual price trajectory; (ii) if the components of the risk process depend on the risk factors: bond-stock yield differential and volatility index.

It is proposed that GBM is supplemented with a risk process, which depends on yields and volatility. Figure 1 displays the path of the bond-stock yield ratio and the VIX. The time covered is from 1990 when VIX was first introduced. In both graphs the mean is plotted to give an indication of stable levels. Notice the diverging paths of the yield ratio and the VIX during the year 2007: rapidly decreasing interest rates in the US have driven down the yield ratio while the equity market implied volatility was increasing and thus anticipating the instability to come.

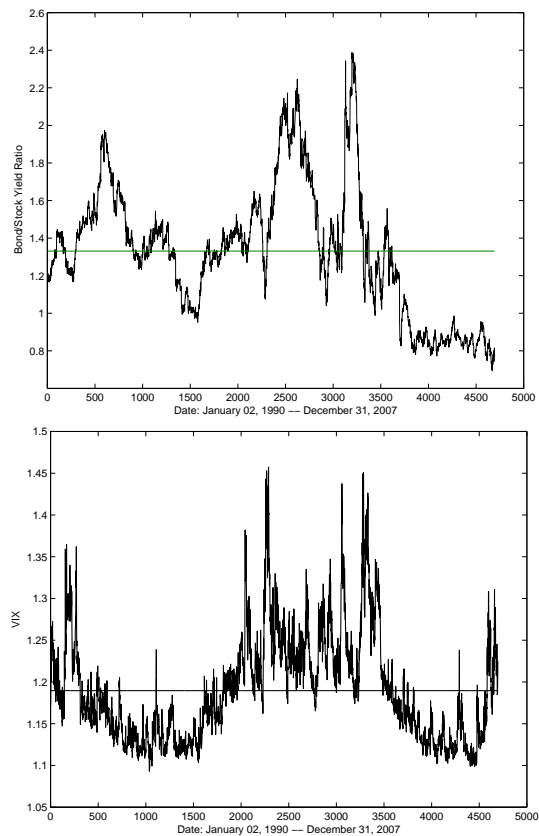


Fig. 1. Risk factors: January 1990 - December 2007

4.1 Single stress measure

The estimated parameters for the model using conditional maximum likelihood with single stress measures generated by separate combinations of the yield ratio

and VIX as risk factors are given in Table 1 and Table 2. The purpose in this analysis is to understand the individual effects of combinations of factors.

Table 1. Diffusion parameter estimates

Parameter	w				
	1.00	0.75	0.50	0.25	1.00
μ_1	0.0000	0.0000	0.0000	0.0000	0.0000
μ_2	0.0006	0.0005	0.0004	0.0003	0.0004
γ_1	0.0004	0.0021	0.0037	0.0014	0.0016
γ_2	-0.0011	-0.0002	-0.0001	0.0001	0.0016
δ_1	0.0042	0.0036	0.0020	0.0039	0.0039
δ_2	0.0094	0.0095	0.0097	0.0095	0.0063

Table 2. Parameter estimates for risk process

Shock	Parameter	w				
		1.00	0.75	0.50	0.25	0.00
UP	θ_{10}	-0.0044	-0.0163	-0.0354	0.0117	-0.0072
	θ_{11}	0.0093	0.0446	0.1181	0.0086	0.0750
	η_1	0.0000	0.0000	0.0000	0.0069	0.0051
	ϕ_1	1.6115	1.4852	1.4561	1.3707	1.2831
	β_1	16.1342	18.7659	23.6526	47.7722	20.7511
DOWN	θ_{20}	-0.0008	0.0112	-0.0016	0.1305	0.0056
	θ_{21}	-0.0007	-0.0416	-0.0241	-0.0591	-0.0717
	η_2	0.0057	0.0057	0.0000	0.0035	0.0055
	ϕ_2	1.4539	1.4330	1.4202	1.3338	1.2738
	β_2	9.4330	17.9974	57.0399	94.5792	21.5550

The estimates are somewhat similar across the various combinations. In all cases the dependence of the shock intensity on the stress measure is strong ($\beta_{ik} \gg 1$). The fitted accumulated increments (log-prices) for the models with the combinations of yield ratio and VIX are given in Figure 2. These fits are actually predictions. That is, with the estimated parameters from in-sample data, a backcast/forecast was performed for the entire study 1990 - 2007. Starting from January 1, 1990, the predicted/expected increments were calculated as:

$$\Delta \widetilde{Y}_2(t) = \hat{\mu}_2 + \hat{\theta}_{10} + \hat{\theta}_{11} E(\Delta N_1(\lambda(\hat{\pi}_1(t-1)))) + \hat{\theta}_{20} + \hat{\theta}_{21} E(\Delta N_2(\lambda(\hat{\pi}_2(t-1)))),$$

where $E(\Delta N_i(\lambda(\hat{\pi}_i(t-1))))$ is the probability of an up/down shock in period t calculated from the Poisson distribution with intensity based on the stress in period t .

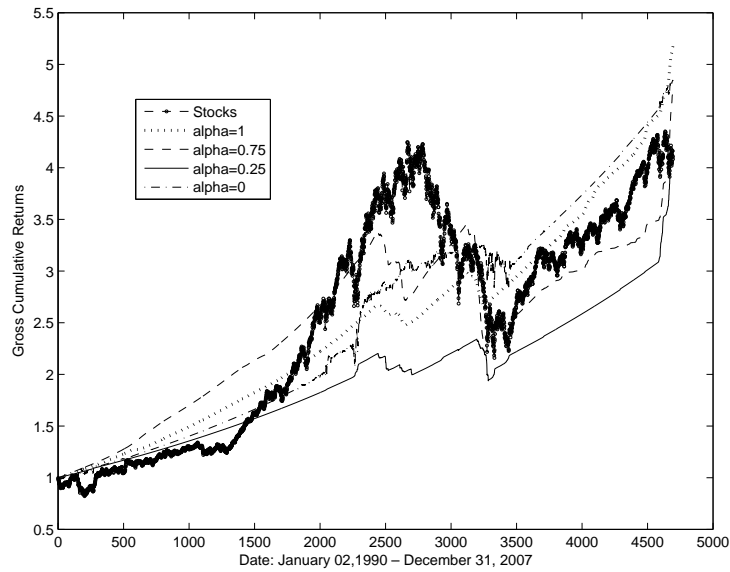


Fig. 2. Fitted prices for the in-sample daily data

In most cases the fits are good, with the risk process capturing the bubble effect of strong growth in stock prices followed by a collapse.

The weights which gave the best fit were: $w = 0.0, 1 - w = 1.0$. These weights were the same for up and down shocks. Although the VIX factor is best overall, the dominance is not uniform over the time period. In some time points, the BSYD ($w = 1.0, 1 - w = 0.0$) is closer to the actual stock price trajectory.

A comparison of the mean squared errors for the models with the yield ratio, the VIX and combinations as risk factors is given in Table 3. The actual SP500 index and the market dynamics implied by the bond yield differential, the VIX and the combinations of the two are reported in Figure 2.

Table 3. Performance comparison: BSYD and VIX combinations

Combination	1.00	0.75	0.50	0.25	0.00
RMSE	0.1897	0.2300	0.3391	0.3050	0.1770
LogLikelihood	34124	34147	34161	34205	35551

These calculations from the full trajectory indicate the dominance of the VIX. There are important points following from the analysis of stock prices with the model containing a diffusion and a risk process based on a single stress measure:

- (i) The inclusion of a marked point process (or risk process) to the diffusion provides a closer match to the dynamics of prices in the US market.

- (ii) When they are considered separately, the bond-stock yield ratio and the volatility index are more significant as risk indicators of the extreme price movements associated with bubbles. The VIX is the single best stress measure.
- (iii) When the stress is defined by combination of the factors, the results are weaker due to an averaging effect, which produces fewer extreme values on the stress measure.

4.2 Multiple stress measures

With the evidence that the factors can be considered individually as stress measures, the multiple stress model is estimated. With both BSYD and VIX included, it is appropriate to work with weekly data, since the BSYD is reported weekly. The best fitting multiple stress model in our analysis is pictured in Figure 3.

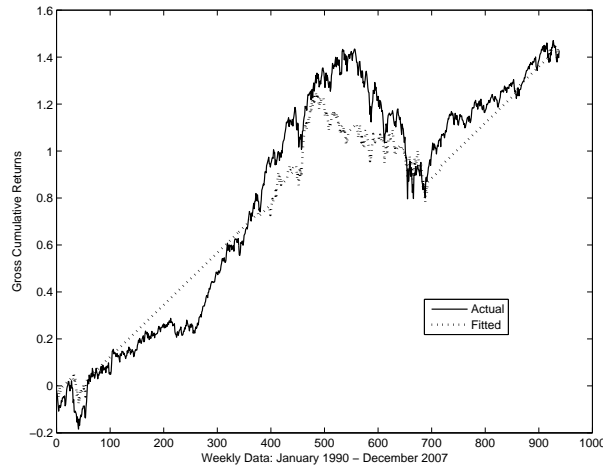


Fig. 3. Fitted prices from multiple stress model for weekly in-sample data

The fit is an improvement over the single stress measure models. The estimated parameters in the fitted model are provided in table 4. For the size estimates, p-values are also reported: (estimate, p-value). The intensity estimates include a confidence interval: (estimate, 95% confidence interval). Tests on standard deviations are not included.

The size parameter estimation included both risk factors. The estimates for the intensity parameters were computed based on the VIX alone, as the size coefficient for the BSYD is negligible. The test results shown in the p-values indicate the statistical significance of the risk factors in the full model. Therefore,

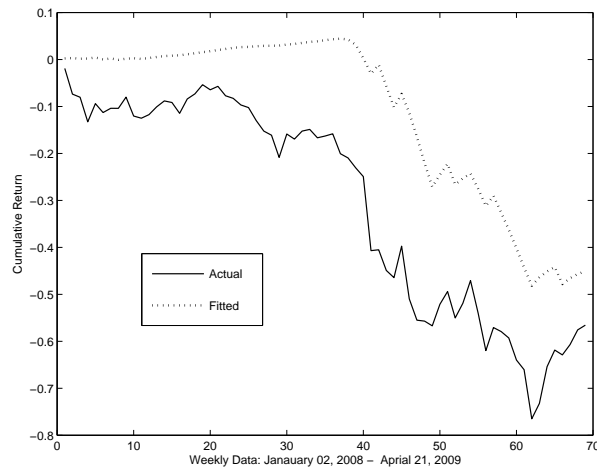
Table 4. Parameter estimates for stress associated components

Market Scenarios	Size Parameters	Intensity Parameters
UP	$\theta_{10} : (-0.0096, 0.1208)$ $\theta_{11} : (0.0136, 0.1448)$ $\theta_{12} : (0.1239, 0.0000)$ $\sigma_1 : (0.0163, *)$	$\phi_1 : [1.3715, (1.3563, 1.3867)]$ $\beta_1 : [33.0178, (25.8816, 42.1152)]$
DOWN	$\theta_{20} : (0.0062, 0.0000)$ $\theta_{21} : (0.0000, 0.9994)$ $\theta_{22} : (-0.1226, 0.0000)$ $\sigma_2 : (0.0182, *)$	$\phi_2 : [1.3160, (1.3058, 1.3262)]$ $\beta_2 : [35.5460, (29.6528, 42.6104)]$

the traditional geometric Brownian motion model is rejected in favor of the risk process with jumps.

4.3 Prediction

The in-sample fitting for the model with the data from 1990 - 2007 is very good, and the obvious issue is the predictability of stock price movements using the fitted model. The estimated model was used to produce weekly forecasts of prices for the period January, 2008 to June 30, 2008. Weekly values for BSYD and VIX are used to produce one week ahead forecasts for shock sizes and probabilities, and the expected shock sizes are added to the Brownian motion increments. The weekly price changes are combined to give the trajectory in Figure 4.

**Fig. 4.** Out-of-sample forecast: January 02, 2008 - April 21, 2009

The forecast follows the pattern of the actual trajectory, although the prediction at the beginning of the period is a bit higher than the actual.

5 Market implications

We have presented a novel approach to study short and medium term equity market returns employing a general random process with a discontinuous component driven by a set of underlying risk sources. The model development rests on the general view that observable market dynamics are to a certain extent motivated by diverging expectations over forthcoming economic and corporate conditions and that a set of stressed factors may induce a sudden market reversal characterized by a change of the correlation structure in the market. As shown in the previous section this view is strongly supported by the achieved market fits. Both instability factors employed in our study are driven by expectations on forward market movements. The approach does not assume the existence of a theoretical equity market fair value but only a potential relative misvaluation given current equity and bond prices.

The bond-yield differential focuses on expected in-outflows from the equity into the bond market, and vice versa, induced by diverging equity and bond yields, resulting in a measure of over-under valuation of the equity market. It is worth remarking that according to the so called Fed model [12] the default free 10 year rate is adopted as a perpetual discount factor of future earnings. Under these assumptions a market increase (decrease) is determined by increasing earnings expectations and decreasing interest rates and an expansion of the credit available in the economy. Similarly a market adjustment will be induced by a sudden revision of earnings expectations associated with an increase of the term structure of interest rates. The above sequence of events is consistent with a possibly very serious market crisis if positive market returns contribute to improve conditional expectations and a restrictive monetary policy turns out to be insufficient over a prolonged period of time to induce a revision of market expectations. This is exactly the sequence of events that anticipated the 1987 market crisis as well as the 2000 dot com crisis.

The volatility index, generated by possibly the most liquid market in the world, reflects expected changes in market forward volatility. This turns out to follow closely the US market reversals. The evidence is in this case: low implied volatility in the option market, positive credit cycle, positive market expectations and positive equity returns. High implied volatility with respect to a 15 year average, unstable expectations, growing market uncertainty, investors' horizon contractions and sudden market adjustments. The latter is also conditional on recent market performance. Market uncertainty is better captured by the volatility index and the stylized evidence of negative market turns after periods of relatively high market volatility is confirmed. The volatility index appears also to provide an effective mapping between leverage-based growth strategies and future corporate performance. Notably, according to the structural approach to credit risk, high implied volatility at an aggregate level implies increasing credit

spreads in the corporate market, reduced earning expectations and, in presence of high leverage after a prolonged period of market growth, anticipates a market reversal. The recent 2007 credit crisis was accurately captured by the index that started around 11% in January 2007, reaching 25% in mid August 2007, where it still is in June 2008! The correlation with the S&P 500 is remarkable. We have in this case a joint signal of credit crunch and negative stock market performance.

In either case, the inclusion of the risk process with random intensity and shock measure facilitates the definition of a risk premium directly associated with such factors and supports the view that equity markets do on average offer a higher expected return than fixed income instruments.

The risk process, driven by market instability factors, clearly captures the swings in stock price trajectories around the standard diffusive process. This supports the intuition about investor behavior when faced with risk. The addition of the risk process to the price model has corresponding implications for the premium on risk for securities. If the average daily returns on stocks and long bonds are calculated for the 1990 - 2006 period, the results are $\hat{\mu}_2 = 0.00038$, and $\hat{\mu}_1 = 0.00036$, which give annualized rates of 1.10 and 1.09, respectively. On that basis the premium for stocks is negligible. With the addition of the risk process, the diffusive returns are $\tilde{\mu}_2 = 0.00059$ and $\tilde{\mu}_1 = 0.00036$. The annualized returns are 1.16 and 1.09 for stocks and bonds respectively. The volatility in the risk process is a part of the premium and increases it to 7%.

In this respect national equity markets will possibly have associated different instability factors and more importantly different risk premia: the higher the correlation between equity markets, the more homogenous the risk structure of equity premia.

6 Conclusions

This paper studies the nature of jumps in equity returns in financial markets. It is proposed that jumps significantly improve the fitting of equity returns, and furthermore the essential features of jumps are affected by current market conditions. That is, the rate of jumps and the size of jumps are variables depending on certain observable risk factors. The model is tested with data on equity returns and the factors: bond stock yield differential, volatility index. From the results of this study the following conclusions are reached.

1. The addition of non-homogeneous point processes to a diffusion greatly improves the fitting to actual equity returns.
2. Both the intensity of jumps and the size of jumps depend on the risk factors - BSYD, VIX.
3. The VIX and BSYD are somewhat complimentary in that during some periods the dependence on the VIX is more pronounced, while in other periods the dependence on the BSYD is clearer.
4. Because of the complementarity, risk measures based on combinations of the factors average out or smooth the extremes and result in a low frequency of shocks and a poorer fit.

A model which anticipates extreme price movements is a key ingredient in a stochastic control model for decisions on investments in risky assets. As such, the risk process model in this paper is a reasonable foundation for a decision model which controls the risk of large losses.

References

1. Berge, K., Consigli, G., and Ziemba, W.T.: The predictive ability of the bond stock earnings yield differential model. *Journal of Portfolio Management*, Spring (2008) 1–18
2. Buhlmann, H.: *Mathematical methods in risk theory*. Springer, Berlin (2006)
3. Campbell, J.Y, and Taksler, G.B.: Equity volatility and corporate bond yields. *Journal of Finance* **58-6** (2003) 2321–2349
4. Consigli, G., MacLean, L.C., Zhao, Y., and Ziemba, W.T.: The Bond-Stock Yield Differential as Risk Indicator in the Financial Markets. *The Journal of Risk* **11-1** (2008)1-22
5. Consigli, G.: Tail estimation and mean-variance portfolio selection in markets subject to financial instability. *Journal of banking and finance* **26-7** (2002) 1355-1382
6. Dempster M.A.H. Ed.: *Risk management: value at risk and beyond*. Cambridge University Press (2002)
7. Durre, A. and Giot, P.: An international analysis of earnings, stock prices and bond yields. Working paper, European Central Bank, Frankfurt (2005)
8. Embrechts, P., Kluppelberg, C. and Mikosch, T.: *Modelling extremal events for insurance and finance*. Springer - Verlag, Heidelberg (1999)
9. Fernandes, M.: Financial crashes as endogenous jumps: estimation, testing and forecasting. *J. of Economic Dynamics and Control* **30** (2006) 111-141
10. Gabaix, X., Gopikrishnan, P., Pevou, V., and Stanley, H. E.: A theory of power-law distributions in financial markets. *Nature* **423** (2003) 267-270
11. Johannes, M., Kumar, R., and Polson, N. G.: State dependent jump models: how do US equity indices jump. Working paper. Department of Economics, University of Chicago (1999)
12. Koivu, M., Pennanen, T. and Ziemba, W. T.: Cointegration analysis of the Fed model. *Finance Research Letters* **2** (2005) 248-259
13. Lim, G. C., Martin, V. L., and Teo, L. E.: Endogenous jumping and asset price dynamics. *Macroeconomic Dynamics* **2** (1998) 213 - 237
14. MacLean, L.M., Zhao, Y., Consigli, G., and Ziemba, W. T.: Estimating parameters in a pricing model with state dependent shocks. in *Handbook of Financial Engineering*, P.M. Pardalos, M.Doumpos and C. Zopounidis (Eds). Springer-Verlag (2008)
15. MacLean, L.M., Foster, M., and Ziemba, W.T.: Covariance complexity and rates of return on assets. *J. of Banking and Finance* **31-11** (2007) 3503 - 3523
16. Merton, R.C.: On the pricing of corporate debt: The risk structure of interest rates, *Journal of Finance* **29** (1974) 449 - 470
17. Pflug, G.Ch. and Romisch, W.: *Modeling, Measuring and Managing Risk*. World Scientific Publisher (2007)
18. Shiller, R.J.: *Irrational Exuberance*, Princeton University Press (2005)
19. Stamatis, D. H.: *Six Sigma and Beyond*, CRC Press (2003)
20. Szego G. Ed.: *Risk Measures for the 21st Century*. John Wiley and Sons Publishers (2004)

21. Ziemba, W. T.: *The Stochastic Programming Approach to Asset, Liability and Wealth Management*. AMIR (2003)
22. Ziemba, W. T. and Schwartz, S. L.: *Invest Japan*. Probus Publishers (1991)
23. Ziemba R. E. S. and Ziemba, W. T.: *Scenarios for Risk Management and Global Investment Strategies*. Wiley (2007)