A bisection algorithm for fuzzy quadratic optimal control problems

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Abstract. In this paper we propose an iterative method to solve an optimal control problem, with fuzzy target and constraints. A solving method is developed for such type of problem, under quite general hypotheses. The algorithm is developed in such a way as to satisfy *as best as possible* both the target function and all the constraints. It consists of an iterative procedure, which modifies the admissible region with the aim to increase at each step the global performance. Even if the procedure is quite general, the algorithm can be applied only if a method exists to solve a crisp parametric sub-problem obtained by the original one. This is the case for a quadratic-linear target function with linear constraints, for which some well established solvable methods exist for the crisp associated sub-problem. The algorithm is particularized to this case, and a numerical test is proposed, showing the quick convergence to the optimal solution.

Keywords. Fuzzy programming, quadratic optimal control.

M.S.C. classification. 90C70.
J.E.L. classification. C61.

1 Introduction

This paper deals with fuzzy optimal control problem (FOCP). The formulation of a FOCP combines both elements of *vagueness* or *ambiguity* in the target or in the constraints and the solving methods for optimal control problems. Given the theoretical difficulties that arise in the solution methods for optimal control problems, few fuzzy extensions were proposed for such cases. We mention only few contribution in the field of optimal control and dynamic programming problems; see [9], [15], [18], [20], [22], [23], [24], [26].

In the specialized literature, fuzzy optimization problems are often classified into *vagueness* or *ambiguity* problems, see [11]. In vagueness problems, *fuzzy* goals and constraints are considered, and usually a *satisfaction* degree is obtained both for the target function and for the constraints; see [2] and [33]. Conversely, in ambiguity problems some of the coefficients are *fuzzy numbers*. Some example of the latter case can be found in [5], [13], [19], [21]. In this paper we shall focus the attention on problems with vagueness, sometimes known as *possibilistic*

optimization problems. In this case, it is required that both the target function and the constraints satisfy as much as possible some required performances, represented by fuzzy numbers, each of them defined by a suitable *membership functions*. From this point of view, they are similar to *goal* programming. Subsequently, the values of the membership functions are aggregated by the *mimimum* triangular norm (*t-norm* MIN), see [32], obtaining the best *compromise* solution.

In so doing, the optimal solution represents the best compromise among the satisfaction degrees of both the objective function and the constraints. In this direction, many contribution appeared in the past, and the interested reader can refer to the surveys presented in [11], [13]. This approach can be particularly useful for real world decision problems, where objectives and constraints are expressed in a heuristic way.
In what follows, an iterative algorithm for fuzzy optimization problems is proposed and applied to a particular case of optimal control problem. In particular, the original problem is decomposed into a set of crisp sub-problems, each of them depending on a parameter λ. The sub-problems are subsequently solved, and the value of the parameter is adjusted at each iteration, in such a way as to converge to the optimal solution. Thus, this iterative approach requires a solving method for the related parametric crisp sub-problems. It can be found that, under some quite general hypotheses, the algorithm converges to the optimal solution in a finite number of steps; see the Propositions 1 and 2 in Section 3.

A wide class of problems that can be solved in this way is the class of quadratic-linear static and dynamic problems with linear constraints. Let us remark that the class of the quadratic optimization problems covers a lot of practical problems in the real world, which justifies the great effort put in the past to develop quick and efficient numerical algorithm.

The paper is organized as follows. Section 2 describes the fuzzy optimization problem. The algorithm is described in Section 3 together with some theoretical result. Section 4 consider the quadratic-linear optimal control problem. Finally, in Section 5 a test simulation is proposed, showing the quick convergence of the proposed algorithm to the optimal compromise solution.

2   Fuzzy optimization problem

Consider the following mathematical programming problem:

$$\begin{cases} \min_{x \in X} f(x) \\ g_i(x) \geq 0 \end{cases} \tag{1}$$

with $x \in R^n, g_i : X \to R, i = 1,..,m$, and $f(x)$ is convex function. Let U, the admissible region of problem (1), be a convex set. The problem (1) can be extended to a *possibilistic* optimization problem, where the borders that differentiate satisfactory from unsatisfactory regions are not *rigid* thresholds, but are represented by suitable fuzzy numbers. To this aim, the objective function and the constraints need to be intended in fuzzy sense, see [30], [31], [32]. That is, an optimizing solution has to satisfy as most as possible both the target and the

constraints, namely to maximize the *minimum* degrees of the target function and the ones of all the constraints[1].

Using a symbology widely applied in the field of fuzzy optimization, the fuzzy version of the problem (1) can be written as:

$$\begin{cases} \widetilde{\min}_{x \in X} f(x) \\ g_i(x) \widetilde{\geq} 0 \end{cases} \tag{2}$$

In what follows, such a problem will be referred as FMP (Fuzzy Mathematical Programming problem). Let the membership functions $\mu_0(z)$, $\mu_i(z)$ represent the satisfaction degrees of the target and of the constraints respectively. The fuzzy mathematical problem (2) can be converted into the *crisp* non linear problem:

$$\max_{x \in X} C(x) \tag{3}$$

where $C(x)$ represents the global satisfaction degree:

$$C(x) = \min\{\nu_0(x), \nu_1(x), ..., \nu_m(x)\} \tag{4}$$

and:

$$\nu_0(x) = \mu_0(f(x)), \nu_i(x) = \mu_i(g(x)) \tag{5}$$

for $i = 1, .., m$. This problem is equivalent to the following one, in the space R^{n+1}, see [30]:

$$\begin{cases} \max_{x \in X, \lambda \in [0,1]} \lambda \\ \nu_i(x) \geq \lambda, i = 0, ..., m \end{cases} \tag{6}$$

The satisfaction degrees assigned to each constraint and to the target function are usually represented by continuous and (almost everywhere) differentiable monotonic fuzzy numbers. In particular, $\mu_i(z) : R \to [0, 1]$, for $i = 1, .., m$ are increasing functions, and $\mu_0(z) : R \to [0, 1]$ is a decreasing function. The following piecewise linear functions are two of the most commonly used for monotonic membership functions (S-type and Z-type fuzzy numbers respectively, see [29]):

$$\mu_0(z) = \begin{cases} 1, & z \leq p_0 \\ \frac{z - c_0}{p_0 - c_0}, & p_0 < z \leq c_0 \\ 0, & z > c_0 \end{cases} \tag{7}$$

$$\mu_i(z) = \begin{cases} 0, & z \leq p_i \\ \frac{z - p_i}{c_i - p_i}, & p_i < z \leq c_i \\ 1, & z > c_i \end{cases} \tag{8}$$

with $p_i < c_i, i = 1, .., m$. Many methods were proposed to solve the problem (6); see the quoted references.

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[1] Besides the *minimum* t-norm, other aggregation operators could be used to compute the global satisfaction degree, see [16], but the minimum operator is commonly used.

In this paper an iterative algorithm is presented, partially following the approaches proposed by [28] and [25] to the linear programming problem. The method is based on an iterative algorithm, that requires to compute the solution of a parametric crisp sub-problem.

3   An iterative algorithm for FMP problems

In most cases, the non linear problem (6) can be difficult to solve analytically. To this aim, an alternative method is proposed, based on an iterative procedure, under the hypothesis that an associated crisp sub-problem can be solved.

First of all, consider the following parametric problem P_λ, $\forall \lambda \in [0, 1]$:

$$\begin{cases} \min_{x \in X} f(x) \\ \nu_i(x) \geq \lambda, i = 1, .., m \end{cases} \tag{9}$$

Let be Ω_λ the admissible region of the problem (9) and x_λ the solution of the problem P_λ. Note that the admissible region of the problem (9) is included in the admissible region of the following unconstrained problem P_f (*unconstrained* in the sense that the parametric constraints $\nu_i(x) \geq \lambda$ are not included):

$$\max_{x \in X} \nu_o(x) \tag{10}$$

If (10) has a solution x_f such that $\nu_0(x_f) \leq \min_i \{\nu_i(x_f)\}$, then such solution cannot be ameliorated and $C(x_f) = \nu_0(x_f)$.

Note that the original FMP problem loses its interest if the problem P_0 does not admit the global minimum, as showed by the following example. As a matter of fact, consider the FMP problem:

$$\begin{cases} \tilde{\min}_x \frac{1}{x} \\ x \tilde{\geq} 3 \end{cases} \tag{11}$$

where:

$$\mu_0(z) = e^{-z^2} \tag{12}$$

and:

$$\mu_1(z) = \begin{cases} 1, & 3 \leq z \\ z - 2, & 2 \leq z < 3 \\ 0, & z < 2 \end{cases} \tag{13}$$

We can write:

$$\nu_0(x) = e^{-\frac{1}{x^2}} \tag{14}$$

$$\nu_1(x) = \begin{cases} 1, & 3 \leq x \\ x - 2, & 2 \leq x < 3 \\ 0, & x < 2 \end{cases} \tag{15}$$

In this case, both the FMP problem (11) and its unconstrained related sub-problem (10) have no solution, as it can be easily checked. Really, the problem P_λ has no solution, $\forall \lambda \in [0, 1]$. As a matter of fact, the satisfaction degree $C(x) = \min\{\nu_0(x), \nu_1(x)\}$ becomes, for $x \geq 3$: $C(x) = \nu_0(x) = e^{-\frac{1}{x^2}}$, since $\nu_1(x) = 1, \forall x \geq 3$, and its minimum does not exist.

However suppose that the membership function (12) is changed as follows:

$$\mu_0(z) = \begin{cases} 1, & z < 2 \\ \frac{4-z}{2}, & 2 \leq z < 4 \\ 0, & z \geq 4 \end{cases} \qquad (16)$$

the optimal solution now exists, and it is given by all the points of the unbounded interval $[3, +\infty)$.

To avoid similar meaningless cases, we suppose that $\forall i = 0, .., m, \exists c_i$ such that $\mu_i(z) = 1, \forall z \geq c_i$ or $\forall z \leq c_i$; this implies that the target function and each constraints are completely satisfied if a threshold is reached, as in (7), (8). Moreover, we require that $\{x \in X : \nu_0(x) = 1\} \neq \emptyset$, and that $\nu_i(x)$ admits a minimum, $i = 1, .., m$. From a practical point of view, those hypotheses are not serious limitations.

Let $I_\lambda(f)$ be the *level set* of a given function $f(x) : R^n \to R, x \in X$:

$$I_\lambda(f) = \{x \in R^n : f(x) \geq \lambda\} \qquad (17)$$

We can now enunciate the following Propositions 1 and 2.

Proposition 1. *For each* $\lambda \in [0, 1], \lambda_1 \geq \lambda_2$ *implies* $\Omega_{\lambda_1} \subseteq \Omega_{\lambda_2}$.

Proof.   If $\lambda_1 \geq \lambda_2$, then $\nu_i(x) \geq \lambda_1$ implies $\nu_i(x) \geq \lambda_2$ and therefore $\Omega_{\lambda_1} = \{x \in X : \nu_i(x) \geq \lambda_1, i = 1, , m\} \subseteq \{x \in X : \nu_i(x) \geq \lambda_2, i = 1, .., m\} = \Omega_{\lambda_2}$. ⋄

Proposition 2. (Necessary optimality condition). *The problem (6) admits a global optimal solution* $\overline{x}$, *in correspondence to a value* $\overline{\lambda} \in [0, 1]$. *Furthermore,* $\nu_i(\overline{x}) \geq \overline{\lambda}, \forall i = 0, 1, ..., m$, *and* $\exists i \in \{1, ..., m\} : \nu_i(\overline{x}) = \nu_0(\overline{x}) = \overline{\lambda}$.

Proof. $A \supseteq B$ implies $max\{r(x) : x \in A\} \geq max\{r(x) : x \in B\}$. Putting $A = \Omega_{\lambda_1}, B = \Omega_{\lambda_2}$, and $r(x) = \nu_0(x)$, on the basis of Proposition 1, we get that $\lambda_1 > \lambda_2$ implies:

$$max\{\nu_0(x) : x \in \Omega_{\lambda_2}\} \geq max\{\nu_0(x) : x \in \Omega_{\lambda_1}\} \qquad (18)$$

Let $x*$ be the solution point of problem (6) and suppose that nevertheless $\nu_0(x^*) > min_i\{\nu_i(x^*)\}$. Due to the continuity of $\nu_0(x)$ and $\nu_i(x), i = 1, ..., m$, also $t(\lambda) = max\{\nu_0(x) : x \in \Omega_\lambda\}$ is a continuous function, and therefore, for (18), it exists a point x^{**} such that $\nu_0(x^*) \geq \nu_0(x^{**}) > min_i\{\nu_i(x^*)\}$ and $\nu_0(x^{**}) = t(\lambda') = max\{\nu_0(x : x \in \Omega_{\lambda'}\}$ with $\nu_0(x^*) > \lambda' > min_i\{\nu_i(x^*)\}$. Being $min\{\nu_i(x^{**})\} > min\{\nu_i(x^*)\}, i = 0, 1, .., m$, we conclude that if $\nu_0(x^*) > min_i\{\nu_i(x^*)\}$ then x^* cannot be the optimal solutions. Analogous proof holds if

$\nu_0(x^*) < min_i\{\nu_i(x^*)\}$. Therefore, if x^* is the optimal solution, then $\nu_0(x^*) = min_i\{\nu_i(x^*)\}$. $\diamond$

Note that the condition stated by the above Proposition 2 is only necessary. In fact, it is very easy to define a function that in every point of its domain satisfies such a condition, even if all those points are not minimizer ones. Referring to (7), (8), this is the case of the function $f(x) : R \to R$, with $m = 1$: $f(x) = \frac{p_0 - c_0}{c_1 - p_1}(x - p_1) + c_0$, which satisfies $\nu_0(x) = \nu_1(x)$ for $p_1 \leq x \leq c_1$, even if no one of these points is a minimizer.

The sufficient condition implies the Pareto optimality for each admissible direction. If we define $J(\lambda) = \{j \in 1, ..., m : \nu_j(\bar{x}) = \lambda\}$, the sufficiency condition requires that $\forall v \in R^n$, with $\| v \| = 1, \exists \epsilon > 0$ so that, if $(\bar{x} + \delta v) \in X$ with $0 \leq \delta \leq \epsilon$, at least one of the following two conditions be satisfied:

a) $f(\bar{x} + \delta v) \geq f(\bar{x})$

or:

b) $\exists i \in J(\lambda) : \nu_i(\bar{x} + \delta v) \leq \nu_i(\bar{x})$.

Obviously, the condition a) implies $\nu_0(\bar{x} + \delta v) \leq \nu_0(\bar{x})$. Moreover, if $\mu_i(z)$, $\mu_0(z)$, $f(x)$, $g_i(x)$ are differentiable in $\bar{x}$, from the formulated hypotheses it follows $\mu_0' \leq 0$ and $\mu_i' \geq 0$, and the two above conditions become:

a.1) $\frac{\partial \nu_0(\bar{x})}{\partial v} \geq 0$, that is $v^T \cdot \nabla \nu_0(\bar{x}) = \mu_0'[f(\bar{x})] \cdot v^T \cdot \nabla f(\bar{x}) \geq 0$, from which $v^T \cdot \nabla f(\bar{x}) \leq 0$

a.2) $\exists i \in J(\lambda) : \frac{\partial \nu_i(\bar{x})}{\partial v} \leq 0$, that is $v^T \cdot \nabla \nu_i(\bar{x}) = \mu_i'[g_i(\bar{x})] \cdot v^T \cdot \nabla g_i[\bar{x}]] \leq 0$, from which $v^T \cdot \nabla \nu_i(g_i(\bar{x})) \leq 0$.

The optimization algorithm is based on Propositions 1 and 2. First of all, suppose that an algorithm exist to solve the parametric problem $P_\lambda, \forall \lambda \in [0, 1]$. Let x_f, x_λ be the values of the solution of the unconstrained problems P_f, (10), and of the sub-problem P_λ, (9), respectively. The algorithm modifies at each iteration the value of λ in such a way as to increase the value of the satisfaction degree.

Then, if the hypotheses of Proposition 2 are satisfied, the following bisection algorithm can be applied to solve the FMP problem.

Bisection algorithm

a) solve the unconstrained problem P_f: $x_f = \text{argmax}_x \nu_0(x)$ (given the stated hypotheses, the existence of a solution is guaranteed). Being $\nu_0(x_f) = 1$ by hypothesis, compute the value $\bar{\nu}(x_f) = \min\{\nu_1(x_f), .., \nu_m(x_f)\}$; if $\nu_0(x_f) \leq \bar{\nu}(x_f)$ then stop, and the solution is optimal with satisfaction degree $C(x_0) = \nu_0(x_f) = 1$, and cannot be ameliorated; otherwise $(\bar{\nu}(x_f) < \nu_0(x_f))$, set $\lambda_{min} = \bar{\nu}(x_f)$, $\lambda_{max} = \nu_0(x_f)$, $\lambda = \frac{\lambda_{min} + \lambda_{max}}{2}$

b) solve the parametric problem P_λ, (9), and compute the values $\nu_0(x_\lambda)$, $\overline{\nu}(x_\lambda) = \min\{\nu_1(x_\lambda), .., \nu_m(x_\lambda)\}$. If $|\nu_0(x_\lambda) - \overline{\nu}(x_\lambda)| < \epsilon$, with $\epsilon > 0$, then stop; the optimal solution is reached, with $x^* = x_\lambda$ and satisfaction degree $C(x^*) = \nu_0(x^*) \cong \overline{\nu}(x^*)$, and $\lambda^* = \lambda$. Else:

c) if $\nu_0(x_f) > \overline{\nu}(x_f)$ then set $\lambda \leftarrow \frac{\lambda + \lambda_{max}}{2}$, $\lambda_{min} = \lambda$, goto b). Else ($\nu_0(x_f) < \overline{\nu}(x_f)$):

d) set $\lambda \leftarrow \frac{\lambda + \lambda_{min}}{2}$, $\lambda_{max} = \lambda$, goto b).

Some remarks are in order:

i) The condition $|\nu_0(x_\lambda) - \overline{\nu}(x_\lambda)| < \epsilon$ checks for the equality of the satisfaction degrees for target and constraints, see Proposition 2.

ii) Naturally, if $\lambda^* = 0$, it means that the admissible region of problem (6) is empty.

iii) The algorithm implements a simple dichotomic approach. To avoid undesired instability some checks are necessary, and this justifies the use of λ_{min}, λ_{max} which represents at each iteration the minimum and the maximum value respectively for the satisfaction degree of the constraints. The value of λ for the next iteration cannot be less than λ_{min}, neither greater than λ_{max}. This can happen if at the actual iteration the membership degree of the target suddenly decreased too much, and the simple dichotomic method in this case can produce instability computing a value of λ less than λ_{min}, or greater than λ_{max}. In this case, see steps c) and d), the value of λ is forced to an intermediate value among λ_{min} or λ_{max} and the actual value of λ (usually equal to the satisfaction degree of the constraints, that is $\overline{\nu}(x_\lambda)$).

iv) Propositions 1 and 2 ensure that the algorithm converges in a finite number of steps. In fact, updating the value of λ as in step d), the convergence of the algorithm is assured in at most $log_2 \frac{1}{\epsilon}$ steps [2]. Anyway, more sophisticated algorithms can improve the speed of convergence, see [10], but they are beyond the scope of this paper.

v) Given the hypotheses, in the optimizing point the sufficient condition is ensured, because starting from x_f the algorithm moves toward the (unique) global optimizing point.

Conversely, if $f(x)$ is not a convex function, the algorithm converges only to local optimizing points, for which the previously stated necessary and sufficient conditions are verified, even if they are not in general, global optimizing points.

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[2] Of course a first iteration is necessary to solve the unconstrained problem P_f.

4   The quadratic-linear FOCP problem

The proposed algorithm could be used to solve both linear and non linear FMP and FOCP problems. The solution of an Optimal Control Problem, see [1], implies the research of function in such a way as to minimize a target index, in the most cases expressed into an integral form. In the linear case many algorithms exist, see for instance [3], [17], [27] and the references therein. On the other side, the optimization problem (9) can be difficult to be solved in the non linear case, and the previous algorithm cannot be applied. In some particular cases the crisp parametric sub-problem is a standard programming problem, whose solution can be easily obtained, and then the proposed iterative algorithm can be applied.

This is the case of the quadratic FMP and FOCP problems, say QFMC and QFOCP respectively, both of them widely used in the real world applications. We only mention, in the static case, the maximization of expected returns in portfolio theory, and the quadratic optimization problem that arises in the learning phase of a Support Vector Machine [7], while in the dynamic case the optimal control problems for industrial regulators and for economic policies.

In this Section, the iterative algorithm developed for the general case in the previous Section will be particularized to the QFOCP case. For the sake of notation simplicity, we limit to analyze the *minimum energy* quadratic optimal control problem with linear dynamic and constraints. In this case, the dynamics is represented by a linear crisp equation (with fixed initial state), while the linear constraints are the final condition are represented by fuzzy constraints. Of course, this is not a theoretical limitation, since the general case of quadratic control problem can be treated in the same way. Then the minimum energy QFOCP (2) can be written as:

$$
\begin{cases}
\widetilde{\min}_{u_0,..,u_{\{T-1\}}} J(u_0,..,u_{T-1}) = \frac{1}{2}\sum_{t=1}^{T-1} u_t' Q_t u_t \\
x_{t+1} = A_t x_t + B_t u_t & \text{dynamic equation,} \quad t = 0,..,T-1 \\
x_o = x^0 & \text{initial condition (fixed)} \\
x_T \widetilde{\geq} \bar{x} & \text{final state condition} \\
G_t x_t + H_t u_t \widetilde{\geq} b_t & \text{linear constraints,} \quad t = 1,..,T-1
\end{cases}
\tag{19}
$$

with $x_0, x_t, x_T \in R^n$, $u_t \in R^m$, $Q_t \in R^{m \times m}$, $A_t \in R^{n \times n}$, $B_t \in R^{n \times m}$, $G_t \in R^{r \times n}$, $H_t \in R^{r \times m}$, $b_t \in R^r$.

Moreover, let be assigned the membership functions $\mu_0(z), \mu_T(z), \mu_t(z)$. From (5) if follows that:

$$
\begin{cases}
\nu_0(u_0,..,u_{T-1}) = \mu_0(J(u_0,..,u_{T-1}) \\
\nu_T(x_T) = \mu_T(x_T) \\
\nu_t(x_t, u_t) = \mu_t({}_t x_t + H_t u_t - b_t).
\end{cases}
\tag{20}
$$

In what follows, we suppose that the conditions for the convexity of the target function are satisfied. To this purpose, the reader can refer to [1] for a

complete description of the necessary conditions. In this case, disregarding the linear constraints, a global minimum can be obtained for the crisp problem, using the feedback-feedforward method based on the Riccati equation, see again [1]. Let us now consider the fuzzy problem. The region U is a convex set. From the propositions 1 and 2, and the convexity of the target function, an optimal solution exists and the proposed algorithm can be used. The crisp parametric sub-problem (9) becomes a quadratic-linear problem with linear constraints, and can be easily solved using the same standard techniques based on the Riccati equation.

To this purpose, we suggest, among other methods, the so called *penalty function*, see [5], [14], that, roughly speaking, builds an unconstrained problem adding to the target function some terms, each of them referring to an unsatisfied constraint. Every term is formed by the squared constraint equation $Ax_i - b_i = 0$, multiplied by a positive value (the *penalty*) whose value is increased at each iteration. The unconstrained modified problem is then solved, and if the solution violates some constraint, the relative penalty is augmented, otherwise the algorithm stops (the optimal solution is reached).

Under some general hypotheses, usually satisfied in the quadratic-linear case, this solution coincides with the one of the original constrained problem, see [10]. Since in the quadratic-linear case the constraints are linear, the penalty terms are quadratic, and the modified (penalized) target function remains a quadratic-linear (and convex) function. Then its optimum value can be easily computed, since we refer to an unconstrained problem. It is remarkable that the penalty function method requires an inner loop to reach the optimal solution in the steps a), b) of the QFMP iterative algorithm.

However many other methods exist to solve a linear-quadratic constrained problem, but their description is beyond the scope of this contribution. The reader can refer to the extensive literature on the topic.

Anywise, for what above said, the sub-problem (9) becomes for the quadratic-linear case (19):

$$\begin{cases} \min \frac{1}{2} \sum_{t=1}^{T-1} u_t' Q_t u_t \\ x_{t+1} = A_t x_t + B_t u_t, \quad x_o = x^0 \\ \nu_T(x_t, u_t) \geq \lambda \\ \nu_t(x_T) \geq \lambda \end{cases} \qquad (21)$$

where $\nu_t(x_t, u_t) = \mu_t(G_t x_t + H_t u_t - b_t \geq 0)$, $\nu_T(x_T) = \mu_T(x_T \geq \bar{x})$.

Since $\mu_t(z)$, $\mu_T(z)$ are increasing S-type fuzzy number, the constraints $\nu_t(x_t, u_t) = \mu_t(G_t x_t + H_t u_t - b_t)$ and $\nu_T(x_t, u_t) = \mu_t(x_T)$ can be written as $G_t x_t + H_t u_t - b_t \geq \inf I_\lambda(\mu_t)$ and $x_T \geq \inf I_\lambda(\mu_T)$. Thus the problem P_λ, can be written as the following QFMP with linear constraints:

$$\begin{cases} \min \frac{1}{2} \sum_{t=1}^{T-1} u_t' Q_t u_t \\ x_{t+1} = A_t x_t + B_t u_t, \quad x_o = x^0 \\ G_t x_t + H_t u_t - b_t \geq \inf I_\lambda(\mu_t) \\ x_T \geq \inf I_\lambda(\mu_T) \end{cases} \tag{22}$$

Being satisfied the hypotheses of the Proposition 2, an optimal solution surely exists.

Note that even if the minimum energy QFOCP, the extension to more general type of QFOCP is straightforward.

5 A numerical test for the minimum energy linear QFOCP

The proposed algorithm was tested using the following time-invariant minimum energy QFOCP, with $T = 2$:

$$\begin{cases} \widetilde{\min} \frac{1}{2}(u_o^2 + u_1^2) \\ x_{t+1} = 0.5 x_t + 2 u_t, \, x_0 = 4 \\ x_T \widetilde{\geq} 9 \\ u_t \widetilde{\leq} 2, \qquad\qquad t = 1, 2 \end{cases} \tag{23}$$

where the membership functions of the target function and of the constraints (final state and control variables) are given by:

a) target function:

$$\mu_0(z) = \begin{cases} 1, & z \leq 4 \\ -0.5z + 3, & 4 < z \leq 6 \\ 0, & z > 6 \end{cases} \tag{24}$$

b) final state constraint:

$$\mu_T(z) = \begin{cases} 0, & z \leq 6 \\ \frac{1}{3}z - 2, & 6 < z \leq 9 \\ 1, & z > 9 \end{cases} \tag{25}$$

c) control variables constraints, equal for both the two control variables (so they are not indicized with the subscript t) as in (19)):

$$\mu(u_t) = \begin{cases} 1, & u_t \le 2 \\ -0.5u_t + 2, & 2 < u_t \le 4 \\ 0, & u_t > 4 \end{cases} \qquad (26)$$

with $t = 1, 2$.

Being $z = u_o^2 + u_1^2$, it is $\nu_0(u_0, u_1) = \mu_o(u_0^2 + u_1^2)$, moreover $\nu_T(x_T) = \mu_T(x_T)$, $\nu(u_0) = \mu(u_o)$, $\nu(u_1) = \mu(u_1)$.

The target function is convex, and the problem verifies all the other hypotheses formulated in the previous Sections to guarantee the applicability and the convergence.

At the iteration n. 0, the problem P_f, $\min\frac{1}{2}\{u_o^2 + u_1^2\}$ with optimal solution $u_0 = u_1 = 0$, and $z = 0$ gives $\nu_0(u_0, u_1) = \nu(u_0) = \nu(u_1) = 1$ and $\nu_{x_T} = 1$. Then $\bar{\nu}(u_0, u_1) = 0$ and $\lambda_{min} = 0$, $\lambda_{max} = 1$, thus $\lambda_1 = \frac{\lambda_{min} + \lambda_{max}}{2} = 0.5$, see the first row in Table 1. From (22), (23), and from (24), (25), (26) the problem P_λ can be formulated as:

$$\begin{cases} \min\frac{1}{2}(u_o^2 + u_1^2) \\ x_{t+1} = 0.5x_t + 2u_t, \, x_0 = 4 \\ u_t \le -2\lambda + 4 \\ x_2 \ge 6\lambda + 6 \end{cases} \qquad (27)$$

For the subsequent iteration n.1, with $\lambda = 0.5$, the optimal solution is given by $u_0 = \frac{6}{5}, u_1 = \frac{12}{5}$ with $z = \frac{18}{5}$. Then, being $\nu_0(u_0, u_1) > \bar{\nu}(u_0, u_1)$, it follows that $\lambda_2 = \frac{\lambda_1 + \lambda_{max}}{2} = \frac{0.5 + 1}{2} = \frac{3}{4}$, see the second row in Table 1. The updated value of λ is now used for the iteration n.2, and so on, as presented in each row of the Table 1.

The iterative algorithm was implemented in MatLab language, using the Quadratic Programming routine QP. With the value $\epsilon = 0.01$, the algorithm converges to the solution in six steps.

The obtained results are presented in Table 1, where each rows corresponds to a complete iteration of the procedure. The first column reports the iteration counter, the second and the third ones contains the value of the minimizing solution, u_0, u_1. The fourth column reports the value of the target function at time t, while the following four columns report the membership degrees of the the three constraints and of the objective function. The column 8 reports the minimum of the membership degrees of the constraints, the column 9 and 10 the values of $\lambda_{min}, \lambda_{max}$, finally the last column reports the value of λ which will

be applied in the next iteration, λ_{t+1}.

Table 1. Results of the bisection algorithm

t	λ_t	u_0	u_1	z	ν_T	$\nu(u_0)$	$\nu(u_1)$	ν_0	$\overline{\nu}$	λ_{min}	λ_{max}	λ_{t+1}
0	0	0	0	0	0	1	1	1	0	0	1	0.5
1	0.5	1.2	2.4	3.6	0.5	1	0.8	1	0.5	0.5	-	0.75
2	0.75	1.4	2.8	4.9	0.75	1	0.6	0.55	0.55	-	0.75	0.62
3	0.62	1.3	2.6	4.22	0.62	1	0.7	0.89	0.62	0.62	-	0.685
4	0.685	1.35	2.7	4.54	0.685	1	0.65	0.73	0.65	0.685	-	0.717
5	0.717	1.37	2.75	4.72	0.717	1	0.62	0.64	0.62	0.717	-	0.733
6	0.733	1.39	2.77	4.8	0.733	1	0.61	0.6	0.61	...	...	...

Note that the second constraint, $\nu(u_0)$, is completely satisfied at each iteration; in fact, at each iteration the variable u_0 is inferior than 2. Even if this example is rather simple, nevertheless it exhibits a rapid convergence to the optimal solution. Moreover, observe that the crisp problem, after some straightforward algebraic manipulations, can be put in the following form:

$$\begin{cases} \min \frac{1}{2}(u_o^2 + u_1^2) \\ u_0 \leq 2 \\ u_0 + 2u_1 \geq 8 \end{cases} \tag{28}$$

no solution exists, since the admissible region is clearly empty. As usual in FMP problems, the introduction of membership functions for both the target and the constraints relaxes the crisp constraints, admitting a partial violation of them. Of course, a price needs to be payed, that is, a partial violation has to be accepted. Anywise, in this case where no solution exists for the crisp problem, it is possible to obtain a solution that at least maximize the global decision maker satisfaction. In other cases, even if a crisp optimal solution exist, the fuzzy solution can be better than the crisp one, maximizing the decision maker's *own* global performance represented by the aggregated satisfaction degrees of *both* the target and the constraints.

In the proposed numerical example the optimal satisfaction degree is given by 0.733, a value that can be considered a good compromise solution. Naturally, lower satisfaction degrees become more and more unacceptable; for very low values, the compromise solution can become completely unsatisfactory.

6   Conclusion

This paper proposes a bisection algorithm for the solution of a fuzzy optimal control problem, where the target function and the constraints satisfaction de-

grees are computed by means of suitable fuzzy numbers. The optimal solution can be obtained by the solution of a parametric crisp sub-problem, varying at each iteration the value of the parameter.

Based on some natural hypotheses on the membership functions and on the target and the constraints, the convergence of the algorithm is assured in a finite number of steps. The algorithm was next applied in the case of the *minimum energy* fuzzy quadratic optimal control problem with linear constraints. A simple simulation test showed satisfactory convergence to the optimal solution.

Possible future extensions regard other type of target function, like the linear-fractional case [8].

Acknowledgements

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