Ca' Foscari University of Venice

Department of Economics

Working Paper

Sebastiano Della Lena

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ISSN: 1827-3580
No. 09/WP/2019

Working Papers
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Keywords
Opinion Dynamics in Networks, Non-Bayesian Social Learning, Stubborn Agents, Speed of Convergence

JEL Codes
D83, D85, D72, Z10

Address for correspondence:
Sebastiano Della Lena
Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta S.Giobbe
30121 Venezia - Italy
e-mail: sebastiano.dellalena@unive.it

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Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta San Giobbe
30121 Venice Italy
Fax: ++39 041 2349210

Non-Bayesian Social Learning and the Spread of Misinformation in Networks *

Sebastiano Della Lena †

March 14, 2019

Abstract

People are exposed to a constant flow of information about economic, social and political phenomena; nevertheless, misinformation is ubiquitous in the society. This paper studies the spread of misinformation in a social environment where agents receive new information each period and update their opinions taking into account both their experience and neighborhood's ones. I consider two types of misinformation: permanent and temporary. Permanent misinformation is modeled with the presence of stubborn agents in the network and produces long-run effects on the agents learning process. The distortion induced by stubborn agents in social learning depends on the "updating centrality", a novel centrality measure that identifies the key agents of a social learning process, and generalizes the Katz-Bonacich measure. Conversely, temporary misinformation, represented by shocks of rumors or fake news, has only short-run effects on the opinion dynamics. Results rely on spectral graph theory and show that the consensus among agents is not always a sign of successful learning. In particular, the consensus time is increasing with respect to the "bottleneckedness" of the underlying network, while the learning time is decreasing with respect to agents' reliance on their private signals.

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*I am grateful to Paolo Pin for his invaluable help and guidance throughout this project. I also thank Kostas Bimpikis, Sergio Currarini, Pietro Dindo, Edoardo Gallo, Francesco Nava, Fabrizio Panebianco, Simon Weidenholzer, Thierry Verdier, and conferences participants at ASSET 2018 and NETEF 2018, for useful comments and discussions. All errors are mine. I gratefully acknowledge the hospitality of Bocconi University, where this project was developed.

†Department of Economics, Ca' Foscari University of Venice. E-mail: sebastiano.dellalena@unive.it

1 Introduction

People form their beliefs and opinions about political, economic and social issues through the information they have. Nowadays, each person is exposed to a continuous stream of news about almost every subject. However, since no one has direct access to "the truth" and different pieces of information are dispersed among agents, people interact together and update their beliefs taking into account those of others. Moreover, agents tend to take in consideration others' belief to conform to their peers, or to some role models in society, even if they do not have better information. There is evidence, in fact, that people's opinions and decisions are affected by friends, neighbors or even influencers, such as sports celebrities, fashion bloggers, political leaders or commentators.[1]

Social media play a fundamental role in the agent's social learning; nowadays, 62% of US adults use them as a source of news (Gottfried and Shearer, 2016). Social media, like Facebook and Twitter, allow agents to receive and share a lot of information in a very short time and to have easy access to other's opinions. This, despite leading to faster dissemination of news and faster social learning, leaves the door open to the spread of fake-news and misinformation or, in general, opinion manipulation (Del Vicario et al., 2016).

This paper considers important questions about social learning when agents receive a constant flow of information. Do social learning leads to a consensus among different individuals? Are agents able to effectively aggregate dispersed information about the underlying state of the world? How much room is there for belief manipulation and misinformation? Despite a large amount of data about economic, social and political phenomena disagreements are ubiquitous in society (Acemoglu and Ozdaglar, 2011). For example, people tend to disagree on many phenomena such as climate change, the effect of a flat tax or the guaranteed minimum income on the society, the effect of LGBT adoptions on the offsprings' nurture, or even the genuineness of the first moon landing. Therefore, we can deduce how consensus and learning are not always reached and there is room for indoctrination and the spread of misinformation. Moreover, since the consensus is a necessary but not sufficient condition for learning to take place, there is reason to wonder if social environments that guarantee a faster consensus would lead also to a faster learning.[2]

[1] For example, Coleman et al. (1966) discusses the role of doctors in prescription of new drugs, Reingen et al. (1984), Feick and Price (1987) and Godes and Mayzlin (2004) study the role of influencer marketing in brand choice by consumers, Martin and Bush (2000) asks if role models influence teenagers' purchase intentions and behavior, and Bush et al. (2004) analyzes the influence of sports celebrity on the behavioral intentions of a particular generation. Fainmesser and Galeotti (2018) propose a model of market interactions between influencers, followers, and marketers.

[2] It can be argued that does not exist a truth with respect to some of these examples and it is only a matter of preferences. However, in this paper, talking about consensus and learning, I refer to opinions about "objective"

The main contribution of the paper is to identify key players (i.e. nodes that if targeted are more effective in influencing the steady state opinion dynamics) through a new centrality measure, and to analyze topological features that favor the spread of misinformation in a social learning framework where there is an underlying true state of the world and agents receive a constant flow of information. I study the spread of misinformation on a network composed by a set of agents, who each period receive noisy signals about the true state of the world and update their belief as a convex combination of the Bayesian posterior beliefs and a linear updating of neighborhood's beliefs, as in Jadbabaie et al. (2012).

I consider two types of misinformation: (*i*) **permanent misinformation** that is repeated over time and produces long-run effects, and (*ii*) **temporary misinformation** that is unmasked in the long run and produce only short period effects.

To model permanent misinformation, I assume the presence in the network of **stubborn agents**. A stubborn agent has a fixed opinion and learns neither by experience nor by peers, she acts only to affect the outcome of others' social learning. Stubborn are representative of different types of spreaders of permanent misinformation both in society or social media. In fact, stubborn can be used to model *prominent agents* (e.g. media, firms or even politicians) that systematically disseminate opinions and information to convey consent on themselves or on a particular idea that they support.[3] Examples of prominent agents that spread misinformation are the so-called *climate deniers*, who promote skepticism about the scientific opinion on climate change (Oreskes, 2004). Stubborn can also be used to model *social bots*; social bots are algorithms that exhibit human-like behavior and are often used to repeatedly share factious or even fake news and negatively (or positively) comment to show a disagreement (or a consensus) higher than the true one. [4]

facts, that do not depend on personal or social preferences. E.g. we cannot say if a policy is "good" or "bad", since it depends on preferences, but, in principle, it is possible to assess its objective effect in a particular economy. Thus, it is important to note that the "truth" is assumed to objectively exist, even if only in a metaphysical realm.

[3]They can even represent agents who have incentives aligned with them; for example, influencers targeted by some firms or political party to support a particular product or policy. Among theoretical results, Galeotti and Goyal (2009) provides a model about strategic diffusion through influencers, while Fainmesser and Galeotti (2015) and Fainmesser and Galeotti (2016) studies influencer marketing in both monopolistic and oligopolistic framework.

[4]It was estimated (Varol et al., 2017) that the share of bots is between 9% and 15% of the total users active on Twitter; in the same way, many Facebook's users are "fakes". Social bots are used to spread fake news stories to influence political debates (Ratkiewicz et al., 2011), manipulate the stock market (Ferrara et al., 2016), and spread conspiracy theories (Bessi et al., 2015), among others. Moreover, Silverman (2016) shows that fake news stories are among the most shared on Facebook.

The results show that having stubborn agents in the network is enough to prevent the consensus, and thus the learning, to be reached. As in classical naive learning model based on linear updating of others' beliefs (DeGroot, 1974; Golub and Jackson, 2010, among others), the steady state agents' opinions depends on stubborn opinions and their centrality; but as in our model agents receive a constant flow of information, the steady state agents' opinions depends also on the true state of the world and on agents' reliance on their private signals (self-weights). In deriving results a new centrality measure is obtained, the **updating centrality** (UC). The updating centrality identifies the key agent in a social learning process and does not depends only on the topology of the underlying network, but positively depends on the steady state beliefs' precision. According to the updating centrality, agents that end up having less sharp opinions at the steady state are less central than others, *ceteris paribus*. The important intuition beyond the concept of updating centrality is that different agents' social learning processes lead to different relevant centrality measures. For example, if agents are able to recall all their past signals, the updating centrality coincides with the Katz-Bonacich centrality of the network without self-weight. The functional form of UC does not depends on agents' distributions of beliefs. Moreover, it is important to underline that the paper presents the updating centrality in the analysis of the diffusion of permanent misinformation, but the concept of UC is general and does not depend on the presence of stubborn agents in the network.

I conclude the analysis of permanent misinformation discussing the optimal strategy of a farsighted monopolistic "sophisticated" stubborn who knows the true state of the world and wants to minimize the distance between the steady state opinion vector and his own position. The stubborn faces a quadratic lying cost increasing in the distance between her declaration and the truth. The main finding is a threshold value for the cost below which the stubborn declares an opinion more extreme than her true opinion. Moreover, results show that the higher the cost of lying the more dangerous a naive stubborn is with respect a sophisticated one.

Due to their use and nature, many among social bots are not credible and fake news are often unmasked in the long run, thus they only have a short period effect in the opinion dynamics; for this reason, I conclude the paper studying the diffusion of temporary misinformation.The analysis about temporary misinformation can be thought as the study of an exogenous shock which temporarily moves opinions away from the steady state; possible examples of this type of misinformation are the diffusion of unreliable fake news in social media or rumors in a social circle. To understand the effect of temporary misinformation, I study the speed of convergence of the social learning process in the nearby of steady state without any stubborn agent. I study both the **consensus time** (CT) and the **learning time** (LT). The consensus time indicates the time of interaction that agents need to reach very similar opinions. The learning time indicates the time of interaction that agents need to reach opinions very close to the true one. I prove that CT and LT in our framework are generally different. Our results are based on spectral graph theory techniques. In particular, using Perron-Frobenius theorem and Cheeger's inequality (Chung, 1996; Cheeger, 1969) I show that the consensus time (in line with previous literature) positively depends through the second largest eigenvalue on the "bottleneckedness", and thus the homophily, of the underlying network. On the other hand, the learning time mainly depends on the strength of weights that agents give to their private signals and, surprisingly, it may not decrease as you decrease the homophily, or in general the"bottleneckedness", of the network.[5] This is due to the fact that, if the level of private information is different across agents, the learning of better informed agents is slowed down by others and this may reduce the speed of learning of the whole society. Technically speaking, the learning time is proportional to the first eigenvalue of the adjacency matrix without self-weight which, after some manipulations, can be shown as negatively related to agents reliance of private signals.

The article is organized as follows. In Section 1.1 I offer a brief literature review. Section 2 lays out the formal framework of the model where I present, and microfound, agents' updating rule. Section 3 studies the effect of permanent misinformation introducing the presence of stubborn agents in the network. Section 3.1 characterizes the steady state opinions' vector and defines the concept of updating centrality. Section 3.2 is devoted to comparative statics and in particular to measure the marginal distortion due to an outgoing link from a stubborn. Section 3.3 studies the optimal opinion to declare for a monopolistic stubborn who want to affect the opinion dynamics. Section 4 studies temporary misinformation discussing the speed of convergence and in particular consensus and learning time showing when they coincide or differ and why. Section 5 concludes.

1.1 Literature Review

The main purpose of this project is to create a deeper link between the literature on learning in networks and the literature on the optimal targeting of individuals to diffuse (mis)information or opinions in a social network. Moreover, I show how a constant flow of information may seriously change both the steady states opinions' vector and the convergence time with respect to standard DeGroot based models, where agents receive at most one signals at the first period. This paper refers to different streams of literature.

Opinion Dynamics and Learning in Networks The literature about opinion dynamics and learning on networks can be divided into two main approaches: *Bayesian*

[5] Surprisingly with respect to the results of the previous literature.

and *non-Bayesian* learning models. (Golub and Sadler, 2016, for a survey). In particular, the social learning process of this paper belongs to the non-Bayesian stream of literature, which began, and strongly relies, on the famous DeGroot model, DeGroot (1974).[6] Starting from the standard DeGroot model, DeMarzo et al. (2003) make explicit the role of the network, while Golub and Jackson (2010) derive general conditions on the adjacency matrix to ensure the reaching of consensus. Concerning our purpose, the main limit of DeGroot like models is that agents can receive signals about the true state of the world only at the first period and then they update their beliefs aggregating others beliefs, as in Golub and Jackson (2010). In such a case is very easy to manipulate others' opinions and spread misinformation in the network. Therefore, to better analyze the spread of misinformation I consider agents who receive signals about the true state of the world at each period. To model the social learning process, I use an approach similar to Jadbabaie et al. (2012), where agents combine their personal experience and the views of their neighbors as a convex combination of the Bayesian posterior beliefs, given their personal signals, and a linear updating of neighborhood's beliefs.

Behavioral explanation for the Learning Process A contribution of the paper is to use a beauty-contest like utility function (in the spirit of Morris and Shin (2002)) to propose an explanation for the mechanisms behind the specific social learning process used (Jadbabaie et al., 2012). Beauty-contest like utility functions are widely used in the literature about learning and opinion dynamics. For example, Bindel et al. (2015) use a similar utility function show the correspondence with the DeGroot model and compute its the inefficiency (with respect to other non-equilibrium strategies) through the price of anarchy. Buechel et al. (2015) study the opinion dynamics when agents may misrepresent their own opinion by conforming or counter-conforming with their neighbors. In Olcina et al. (2017) the beauty-contest like utility function is used to study the norms' assimilation of ethnic minorities. A similar payoff structure is used also in Bolletta and Pin (2019) where a dynamic process, with co-evolution of both individual opinions and network, is characterized. Molavi et al. (2018), studies the behavioral foundations of non-Bayesian models of learning over social networks under the main behavioral assumption of "imperfect recall" of others' beliefs, showing that, social learning rules have a log-linear form, as long as imperfect recall is the only point of departure from Bayesian rationality. In the end, Dasaratha et al. (2018) study a set of Bayesian agents that learn about a moving target, the main result is that, under incomplete information, a fully Bayesian learning model can be tractable as the standard DeGroot heuristic.

[6] Some paper belonging to the Bayesian learning literature are Bala and Goyal (1998), Gale and Kariv (2003), Rosenberg et al. (2009), Acemoglu et al. (2011), Acemoglu et al. (2014), Lobel and Sadler (2015), Mossel et al. (2015). For a deeper discussion about both Bayesian and non-Bayesian paradigms I refer to the survey Golub and Sadler (2016).

Stubborn Agent To model permanent misinformation, I make use of stubborn agents. Yildiz et al. (2013) use the concept of stubborn agents, firstly proposed by Mobilia (2003) and Mobilia et al. (2007),[7] in a binary opinion dynamics framework. While, models as Grabisch et al. (2017) and Mandel and Venel (2017) study influences and targeting in networks, through stubborn agents, using the DeGroot model for the learning process.

Targeting in Networks Since the position of the stubborn in the network is of primary importance, results refer also to the literature that studies the problem of optimal targeting in networks (Bloch, 2016, for a survey). Papers as Galeotti and Goyal (2009), Candogan et al. (2012) and Fainmesser and Galeotti (2015) study targeting and pricing problem from a monopolistic point of view, while Goyal et al. (2014), Fainmesser and Galeotti (2016) and Bimpikis et al. (2016) deal with the same problem in the competitive case. Our main contribution is to make the first attempt to consider a targeting problem, through stubborn, in a society where agents receive each period signals about the true state of the world, therefore the tension between the learning and the targeting force is very relevant in our problem.

Key Player Important contributions, in economics, to targeting problems stem also from the literature that aims to identify the key agent in a network (Zenou, 2016, for a survey). Among the first and most important results, Ballester et al. (2006) define the *intercentrality* measure, a centrality measure that takes into account not only a player's centrality but also her contribution to others' centrality. While Banerjee et al. (2013) derive a measure of *diffusion centrality* that discriminates between information passing and endorsement. The paper contributes to this literature through the concept of the updating centrality measure, which directly depends on the agents' updating rule.

Speed of Learning The effect of temporary misinformation has only short-run effects, therefore I analyze the speed of learning and the convergence to consensus. Golub and Jackson (2012) examine how the speed of learning of average-based updating processes (as DeGroot) depends on homophily, showing that convergence to a consensus, is slowed by the presence of homophily.[8] Jadbabaie et al. (2013) characterizes the rate of learning in terms of the relative entropy of different agents' signal structures and their eigenvector centralities. Our contribution is to show that, if agents receive signals at each period, which anchor them to the truth, then the learning time can be different from the consensus time. Moreover, LT and is not necessarily related to the homophily level in society.

[7] They use the term zealot instead of stubborn.

[8] In Golub and Jackson (2012), the speed of convergence to a consensus is equivalent to the speed of learning.

2 The Model

In this section, I introduce the baseline model; while in sections 3 and 4, I study the effect of systematic and temporary misinformation on agents' social learning respectively.

The Society The society is represented by a graph $\mathcal{G}(N, \boldsymbol{\mathcal{A}})$, where $N = \{1, 2, ..., n\}$ is the set of finite nodes or agents and $\boldsymbol{\mathcal{A}} \in [0,1]^{N \times N}$ is the matrix that captures the interaction patterns among agents in N. In particular, a_{ij} is the ij-th entry and represents the weight that i gives to agent j, namely how much i listen j in proportion to others agents in N. Each agent $i \in N$ divides her attention between herself and the other agents in N, thus the matrix is row-stochastic and its entries across each row are normalized, $\sum_{j \in N} a_{ij} = 1$. I keep the analysis as general as possible considering a directed network where the interactions can be asymmetric and one-side, so that $a_{ij} >$ while $a_{ji} = 0$.[9] The set of neighbors of player i, namely agents that have a positive influence over i, is denoted by $\mathcal{N}_i := \{j \in N : a_{ij} > 0\}$.

States of Nature and Signals The finite set of possible states of nature is $\Theta \subseteq \mathbb{R}$ where its element θ^* is the true state of the world. Conditional on the true state of the world θ^*, at each time $t \in \mathbb{N}$, each agent i observes a noisy signal $\omega_{i,t}$, and $\boldsymbol{\omega}_t := (\omega_{i,t})_{i \in N}$ denotes the (column) vector with all the realized signals at t as elements. Signals, that a generic i receive during her life, are drawn from a Gaussian distribution with mean $E[\omega_i] = \theta^*$, variance $\sigma^{2\omega_i} > 0$, and precision $\tau^{\omega_i} = \frac{1}{\sigma^{2\omega_i}}$, for all $i \in N$. I also assume that for each agent signals are $i.i.d.$ over time and agents do not need to have any information about signals generation processes.

Agents' Opinion Each agent i, has at each time t a probability distribution (or probabilistic belief) over the possible state of the world $p_{i,t}(\theta) \in \Delta\Theta$. The opinion (belief) of i, at each t, is the first moment of her probability distribution over Θ, $\mu_{i,t} = \int_{\Theta} \theta p_{i,t}(\theta) d\theta$, and $\boldsymbol{\mu}_t := (\mu_{i,t})_{i \in N}$ denotes the (column) vector of opinions at t. While, $\sigma^{2p}_{i,t}$ and $\tau^{p}_{i,t}$ are respectively the variance and the precision of the probability distribution of i over the possible states at t. At time 0, $p_{i,0}(\theta)$ is assumed to be Gaussian for all $i \in N$.

Social Learning I assume that agents observe through communication their neighbors' beliefs. The observed beliefs are used, jointly with private signals received at each period, to update beliefs about the underlying state of the world. The matrix $\boldsymbol{\mathcal{A}}$ describes weights of communication and reliance on their private signals.[10] I say that agents have

[9] We can trivially restrict the analysis to directed networks, where $a_{ij} = a_{ji}$, results do not change.

[10] Notice that I are assuming that individuals update their true opinions in a non-fully rational way. In fact, if individuals were fully rational, they would perfectly account for repetition of information. Empirical evidence strongly suggests that individuals are not fully-rational in these settings. For example, laboratory experiments shows that in both complex networks (Grimm and Mengel, 2014) and also in small social networks with common

imperfect recall (IR) when they take into account only the last signal received in the updating process. On the other hand with **perfect recall** (PR) agents are able to recall and use the whole history of received signals. In this model, agents have alway imperfect recall about others' beliefs treating them as sufficient statistics for the entire history of their observations.[11] While I do not impose any restriction about the recall of past private signals studying both imperfect and perfect recall.

Formally, the updating rule of probabilistic belief for each agent $i \in N$ is assumed to be a convex combination of the Bayesian updating β, and the average probabilistic beliefs of their neighborhood (Jadbabaie et al., 2012)

$$p_{i,t+1}(\theta) = a_{ii}\beta_{i,t+1} + \sum_{j \in \mathcal{N}_i} a_{ij} p_{j,t}(\theta) \tag{1}$$

Where

$$\beta_{i,t+1} = \frac{l(\omega_{i,t+1}|\theta)p_{i,t}(\theta)}{\int_\Theta l(\omega_{i,t+1}\theta)p_{i,t}(\theta)d\theta}$$

is the Bayesian posterior belief at t for agent i and $l(.|\theta^*)$ is the likelihood function that generates signals ω. Elements a_{ii} are the self-reliance of each agent i, while a_{ij} represents how agent i weights j's beliefs.

To study opinion dynamics, in this paper, I focus on the mean of the distribution. At each time t the belief (opinion) of agent i is the first moment of the probability distribution (1), namely

$$\mu_{i,t+1} = a_{ii} \int_\Theta \theta \beta_{i,t+1} d\theta + \sum_{j \in \mathcal{N}_i} a_{ij}\mu_{j,t}. \tag{2}$$

Before to discuss, in the next session, permanent sources of misinformation it is important to provide a behavioral explanation for our particular social learning rule.

Microfoundation There can be many reasons for which agent may aggregate others' beliefs. For example, since agents do not have complete information about the distribution from which signals are drawn, they want to aggregate others' information. Another possibility is that they have preferences to conform to other agents or to specific role models in society. These considerations lead us to offer a micro-foundation for the updating

knowledge (Corazzini et al., 2012), people fail to properly account for repetitions of information.

[11] I refer to Molavi et al. (2018) for a deeper discussion about the implication of imperfect recall of others' belief in social learning. Block et al. (2019) studies the effect of short (or long) memory for problems of learning in games with social comparison.

rule (1) of agents in the society with the following utility function.[12]

$$u_{i,t}\left(p_{i,t}(\theta), p_{-i,t}(\theta)\right) = -\left(p_{i,t}(\theta) - 2a_{ii}\beta_{i,t+1}\right)^2 - \left(p_{i,t}(\theta) - 2\sum_{j\in\mathcal{N}_i} a_{ij}p_{j,t}(\theta)\right)^2 \qquad (3)$$

Solving first order conditions for (3) we find exactly the updating rule in (1).[13]

In next section, I introduce stubborn agents in the network.

3 Permanent Misinformation

In this section, I discuss the presence of a set of stubborn agents $S = \{s_1, s_2, ..., s_m\}$ in the society. Stubborn represent "prominent agents" (as climate deniers, firms, social media, political parties etc.) who have a fixed opinion and repeatedly share factious information or actual misinformation in social media (or social cliques). A generic stubborn $s_s \in S$, is a particular agent that is not affected by the opinion of others and never revises her opinion θ_{s_s}, namely $a_{s_s s_s} = 1$ and $a_{s_s i} = 0$ for all $i \in N$.[14] Let denote with $\boldsymbol{a}_{s_s} := (a_{is_s})_{i\in N}$ the (column) vectors containing all influences of stubborn $s_s \in S$ over non stubborn agents.

As discussed in the previous section, agents may care about stubborn's opinions, in (3), for many reasons. For example, agents may ignore the presence of stubborn in the society, or they may believe that stubborn have information that they do not have or they see stubborn as role models and thus have preferences to conform to them. Thus, equation (3) still holds, an agent i takes into consideration the opinion of stubborn s_s if $s_s \in \mathcal{N}_i$.

With two stubborn (s_1, s_2) in the network, the new adjacency matrix $\boldsymbol{\mathcal{A}}_s$ representing society with stubborn, is defined as[15]

[12]The first appearance of a similar utility function in economics is due to Morris and Shin (2002). I can find it in similar frameworks in Bindel et al. (2015), Olcina et al. (2017), Dasaratha et al. (2018), Bolletta and Pin (2019), among others.

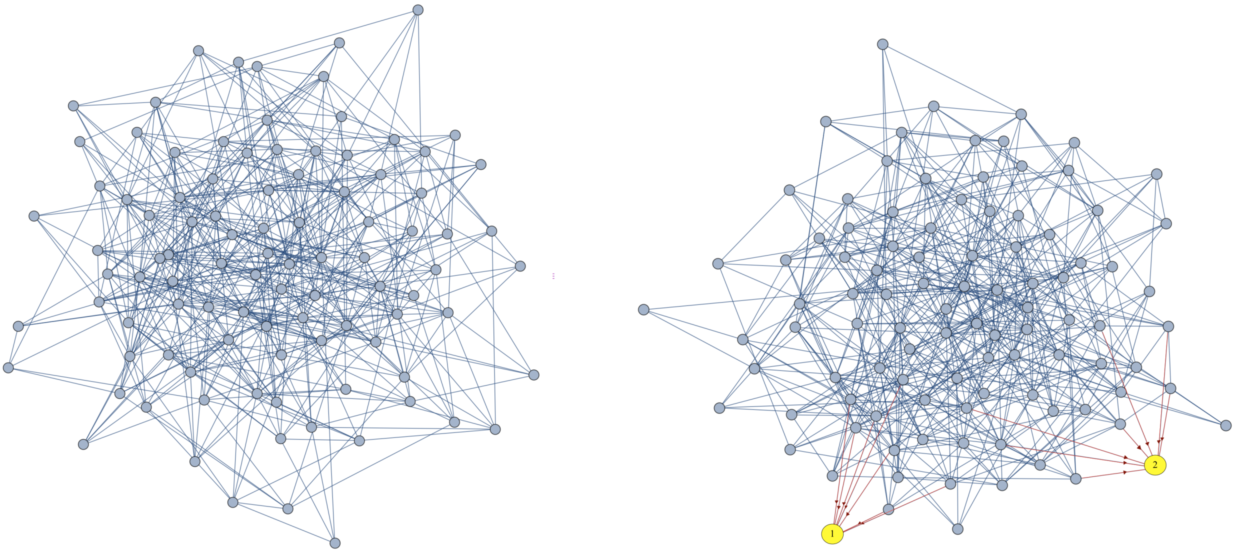
[13]Which is equivalent to Jadbabaie et al. (2012).

[14]Our stubborn agent are the same as the stubborn in Yildiz et al. (2013) and the zealot in Mobilia (2003) and Mobilia et al. (2007).

[15]Here I consider the case with only two stubborn agents, but it can be easily generalized to any number of stubborn.

$$\boldsymbol{\mathcal{A}}_s := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & a_{1s_1} & a_{1s_2} \\ a_{21} & a_{22} & \dots & a_{2n} & a_{2s_1} & a_{s_2} \\ \dots & \dots & & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{c|cc} \boldsymbol{\mathcal{A}} & \multicolumn{2}{c}{\boldsymbol{a}_{s_1}, \boldsymbol{a}_{s_2}} \\ \hline \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{array}\right]$$

Where $\boldsymbol{a}_{s_1}$ and $\boldsymbol{a}_{s_2}$ are the column vectors composed by the weight that each agent i gives to the stubborn. Notice that, with stubborn agents in the society, $\boldsymbol{\mathcal{A}}$ is row-substochastic and $\boldsymbol{\mathcal{A}}_s$ is row-stochastic, such that $\sum_{j \in N} a_{ij} + \sum_{s_s \in S} a_{is_s} = 1$, for each $i \in N$.



(a) (b)

Figure 1: (a) Network with 198 nodes and 486 edges $\mathcal{G}(N, \boldsymbol{\mathcal{A}})$ (b) same network with 2 stubborn agents with 7 link each $\mathcal{G}'(N \cup S, \boldsymbol{\mathcal{A}}_s)$

Stubborn receive always the same signals ω_{s_s} for all $s_s \in S$ with $E[\omega_{s_s}] = \theta_{s_s}$ and zero variances. Since stubborn agents never revise their opinions, neither through social interaction nor through signals, their beliefs are fixed over time, $\mu_{s_s,t+1}(\theta^*) = \mu_{s_s,t}(\theta^*) = \theta_s$.[16]

With stubborns, the updating of non-stubborn agents (2) can be written as

$$\mu_{i,t+1} = a_{ii} \int_{\Theta} \theta \beta_{i,t+1} d\theta + \sum_{j \in N} a_{ij} \mu_{j,t} + \sum_{s_s \in S} a_{is_s} \theta_{s_s} \tag{4}$$

Where the weight that i gives to a generic stubborn ($s_s \in S$) is $a_{is_s} \neq 0$ if and only if

[16] I can provide a micro-foundation for stubborn, but in this case it is trivial.

$s_s \in \mathcal{N}_i$. [17] Therefore, (2) and (4) have exactly the same meaning when $s_s \in \mathcal{N}_i$.

In next section, 3.1, I discuss the effect of stubborn opinions on the steady state opinions' vector of other agents.

3.1 Characterization of Steady State Opinions' Vector

We now characterize the effect of permanent misinformation, modeled as stubborn agents, to the steady state opinion dynamics.

Let now write the opinions' updating rule (4), for the whole society, in matrix form. We decompose the adjacency matrix $\boldsymbol{\mathcal{A}}$ as the sum of the diagonal matrix containing all the weight that agents give to their private signals (self-loops), $\boldsymbol{D} := diag[a_{11}, ...a_{nn}]$, and the adjacency matrix of the network without self-loops, $\boldsymbol{A} := \boldsymbol{\mathcal{A}} - \boldsymbol{D}$. In particular, $\boldsymbol{D}$ represents the weights that agents give to the Bayesian updating and $\boldsymbol{\mathcal{A}}$ the linear (DeGroot) updating. We further define $\bar{\beta}_{i,t} := \int_\Theta \theta \beta_{i,t} d\theta$ as the first moment of the Bayesian posterior, at t, and $\bar{\boldsymbol{\beta}}_t := (\bar{\beta}_{i,t})_{i \in N}$ as the (column) vector containing the all $\bar{\beta}_{i,t}$. Each $\boldsymbol{a}_{s_s}$ is the vector containing all influences of stubborn $s_s \in S$ over agents.

In matrix form the updating of non-stubborn agent is

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{D} \cdot \bar{\boldsymbol{\beta}}_{t+1} + \boldsymbol{A}\boldsymbol{\mu}_t + \sum_{s_s \in S} \boldsymbol{a}_{s_s} \theta_{s_s} \tag{5}$$

Before to characterize the steady state of opinion dynamics, let us define $\boldsymbol{G}$ as the diagonal matrix with γ_i as entries. γ_i depends on the steady state probability distribution, and, in particular, it is increasing in the precision of the probability distribution for i at the steady state, τ_i^p.

Proposition 1 *Given, for all agents $i \in N$, $p_{i,0}(\theta)$ the probability distribution over Θ at $t = 0$ (the prior), and IR of past signals holds, then at the steady state the probabilistic belief, $p_{i,\infty}(\theta)$, converges to and the steady state vector of opinions $\boldsymbol{\mu} := \boldsymbol{\mu}_\infty$ (the first moment of the probability distributions) is*

$$\boldsymbol{\mu} = \boldsymbol{C} \left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + \sum_{s_s \in S} \boldsymbol{a}_{s_s} \theta_{s_s} \right), \tag{6}$$

where

$$\boldsymbol{C} \cdot \mathbf{1} = (\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1} \cdot \mathbf{1} \tag{7}$$

[17] Notice that agents may both not to be aware of the presence of stubborn in the network and can consider them as other agents, or even they may be aware of their presence but they can consider them for other sociological reasons.

is the vector of updating centrality.

Proof. In the Appendix □.

Proposition 1 shows that the steady state opinion vector is a linear combination of the underlying state of the world and stubborn opinions. Moreover, the relative influence that an agent has over opinions' steady state, and thus the weights of the linear combination, depends on the updating centrality (UC) and on the weight that she gives to her private signals, and to stubborn agents, respectively.

This result is particularly relevant with respect to the literature about targeting and the key player in networks (Bloch, 2016; Zenou, 2016, for reviews of these literatures). If a firm (or political party) wants to target agents who receive a constant flow of information to disseminate their message, it should target the more central ones with respect the UC measure, previously defined.

The updating centrality represents the relative influence of each node at the steady state of a particular social learning framework.[18] The UC measure does not satisfy the axiom of *symmetry* of Bloch et al. (2017) and it is particularly interesting because does not depends only on the topology of the underlying network, but positively depends also on the steady state belief's distribution and in particular precision τ_i^p.[19] Thus, both the topological centrality and also the strength of agents' opinions are important. Namely, according to UC, agents that end up to have less sharp opinions at the steady state, *ceteris paribus*, are less central than others. The intuition is straightforward, the more the position of an agent allows her to have a sharper opinion at the steady state (higher precision), the more, if targeted, her opinion remains close to the stubborn opinion. In this way the targeting action results to be more effective. Roughly speaking the steady state belief's precision τ_i^p is a measure of how effectively stubborn can convey her opinion to and through the targeted agent i.

As a particular case I consider the situation in which agents are more rational and are able to recall all past signals.[20]

[18]Notice that the vector of "updating centrality" depends on the updating rules, and priors' distribution. We are going to characterize the "updating centrality" only in the specific case of our model, but, for example, it is trivial to see that for the DeGroot like models the "updating centrality" is nothing but the eigenvector centrality.

[19]The steady state belief's precision of an agents, depend on both from the network structure but also initial belief.

[20]I disregard intermediate cases, where agents recall some of the past signals, since they do not add much to the understanding of social learning.

Corollary 1.1 *Given, for all agents $i \in N$, $p_{i,0}(\theta)$ the probability distribution over Θ at $t = 0$ (the prior), and PR of past signals holds, then at the steady state the probabilistic belief, $p_{i,\infty}(\theta)$, converges and the vector of opinions (the first moment of the probability distributions) is*

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\boldsymbol{D}\boldsymbol{\theta}^* + \sum_{s_s \in S} \boldsymbol{a}_{s_s}\theta_{s_s}\right), \tag{8}$$

where

$$\boldsymbol{C} \cdot \mathbf{1} = (\boldsymbol{I} - \boldsymbol{A})^{-1} \cdot \mathbf{1} \tag{9}$$

is the vector of "updating centrality".

Proof. In the Appendix □.

Proposition 1 and Corollary 1.1 show that the presence of stubborn agents in the network prevents the consensus to be reached at steady state. In this case, since the belief's precision τ_i^p is the same for all agents, the UC depends only on the underlying network and coincides with the Katz-Bonachic (with parameter 1) centrality of the network without self-weight.

As we have seen in Proposition 1 and Corollary 1.1, the underlying state of the word θ^* plays a role in determining the steady state opinions' vector $\boldsymbol{\mu}$. To better present the effect of stubborn and the role of the constant flow of information each period, I provide two examples. In the first one I consider as a benchmark a set of agents that do not listen the received signals and thus update their belief as in a standard DeGroot framework.[21] While, in the second one I consider agents that give positive weight to their private signals.

Example 1 Consider a society composed by four agents, $N = \{1, 2, 3, 4\}$ and two stubborn $s = \{s_1, s_2\}$, with the true state being $\theta^* = 10$ and stubborn's opinions $\theta_{s_1} = 15$ and $\theta_{s_2} = 5$, respectively. Let us assume that all agents have no self-reliance, $a_{ii} = 0$ for all $i \in N$ and the updating rule is the standard DeGroot linear updating model. Consider the social structure described in Figure 2, where the intensity of each link is $\frac{1}{|\mathcal{N}_i|}$, with $i \in \mathcal{N}_i$, for all $i \in N$. Even if the society is wise at $t = 0$, namely $\mu_{i,0} = \theta^* = 10$ for all $i \in N$, the steady state opinions in (a) are 15 for all agents. Namely, if there is only one stubborn in a network where agents linearly update their beliefs without constant signals (as in DeGroot models) the opinion of each agent always converges to the stubborn's opinion. On the other hand, if there is more than one stubborn in the network, as in (b) the steady state opinion vector depends on the centrality of agents connected with stubborn. In particular, $\boldsymbol{\mu}'\mathbf{1}/n$ is 10.238 in (b). ▲

[21]This updating rule is the same used in Golub and Jackson (2010, 2012), in such a case our model with stubborn is comparable to models as Yildiz et al. (2013) or Grabisch et al. (2017).

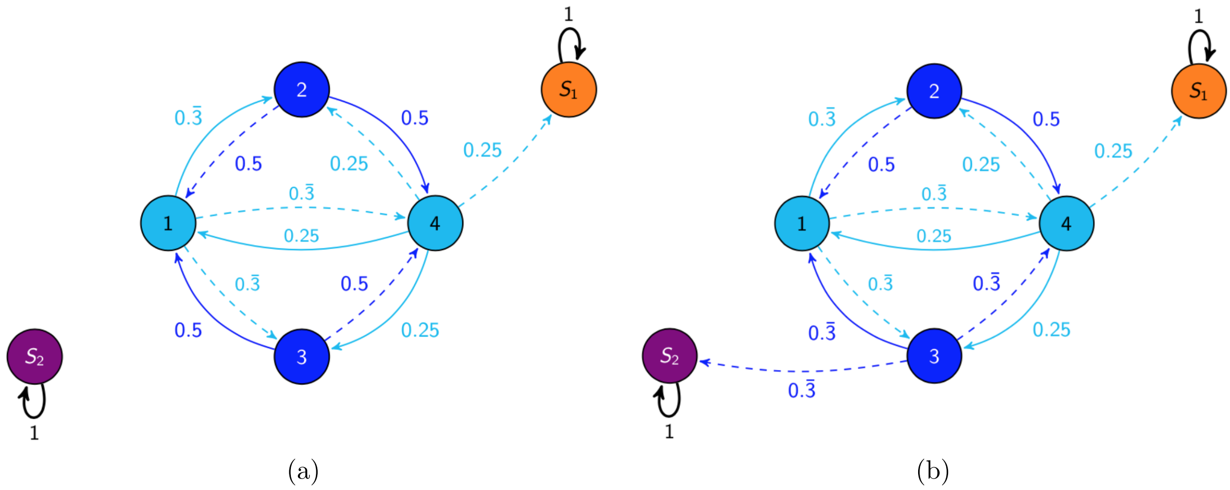


Figure 2: Networks composed by agents who learn in DeGroot fashion, where the intensity of each link of i is $\frac{1}{|\mathcal{N}_i|}$. $\theta^* = 10$, $\theta_{s_1} = 15$, $\theta_{s_2} = 5$. $\boldsymbol{\mu}'\mathbf{1}/n$ is 15 (a) and 10.238 (b).

In the following example, to better understand the importance of the constant flow of new information I consider the same society, state of the world and stubborn as in *Example 1*, but now agents have a positive reliance on their private signals.

Example 2 Consider society composed by four agents, $N = \{1, 2, 3, 4\}$ and two stubborn $s = \{s_1, s_2\}$, with the true state being $\theta^* = 10$ and stubborn's opinions , $\theta_{s_1} = 15$ and $\theta_{s_2} = 5$, respectively. Moreover, agents care about past signals, $a_{ii} > 0$ for all $i \in N$, and PR holds. Consider the social structure described in Figure 3, where the intensity of each link is $\frac{1}{|\mathcal{N}_i|}$, with $i \in \mathcal{N}_i$, for all $i \in N$.

	(a)		(b)		(c)		(d)	
	$\boldsymbol{\mu}$	$\boldsymbol{C}\cdot\mathbf{1}$	$\boldsymbol{\mu}$	$\boldsymbol{C}\cdot\mathbf{1}$	$\boldsymbol{\mu}$	$\boldsymbol{C}\cdot\mathbf{1}$	$\boldsymbol{\mu}$	$\boldsymbol{C}\cdot\mathbf{1}$
1	10	3.6	10.7143	3.22857	9.318	3.273	10.03	2.98
2	10	3.4	10.7143	3.02857	9.545	3.182	10.3	2.88
3	10	3.4	10.7143	3.02857	8.41	2.64	8.97	2.41
4	10	3.6	11.4286	2.857	9.318	3.273	10.86	2.65
	$\frac{1}{4}\boldsymbol{\mu} = 10$		$\frac{1}{4}\boldsymbol{\mu} = 10.893$		$\frac{1}{4}\boldsymbol{\mu} = 9.1477$		$\frac{1}{4}\boldsymbol{\mu} = 10.0414$	

Table 1: Steady state opinions vectors and centralities relative to Example 2, where $\theta^* = 10$, $\theta_{s_1} = 15$, $\theta_{s_2} = 5$.

In Table 1 we can see the centralities and the opinions at the steady states of agents in

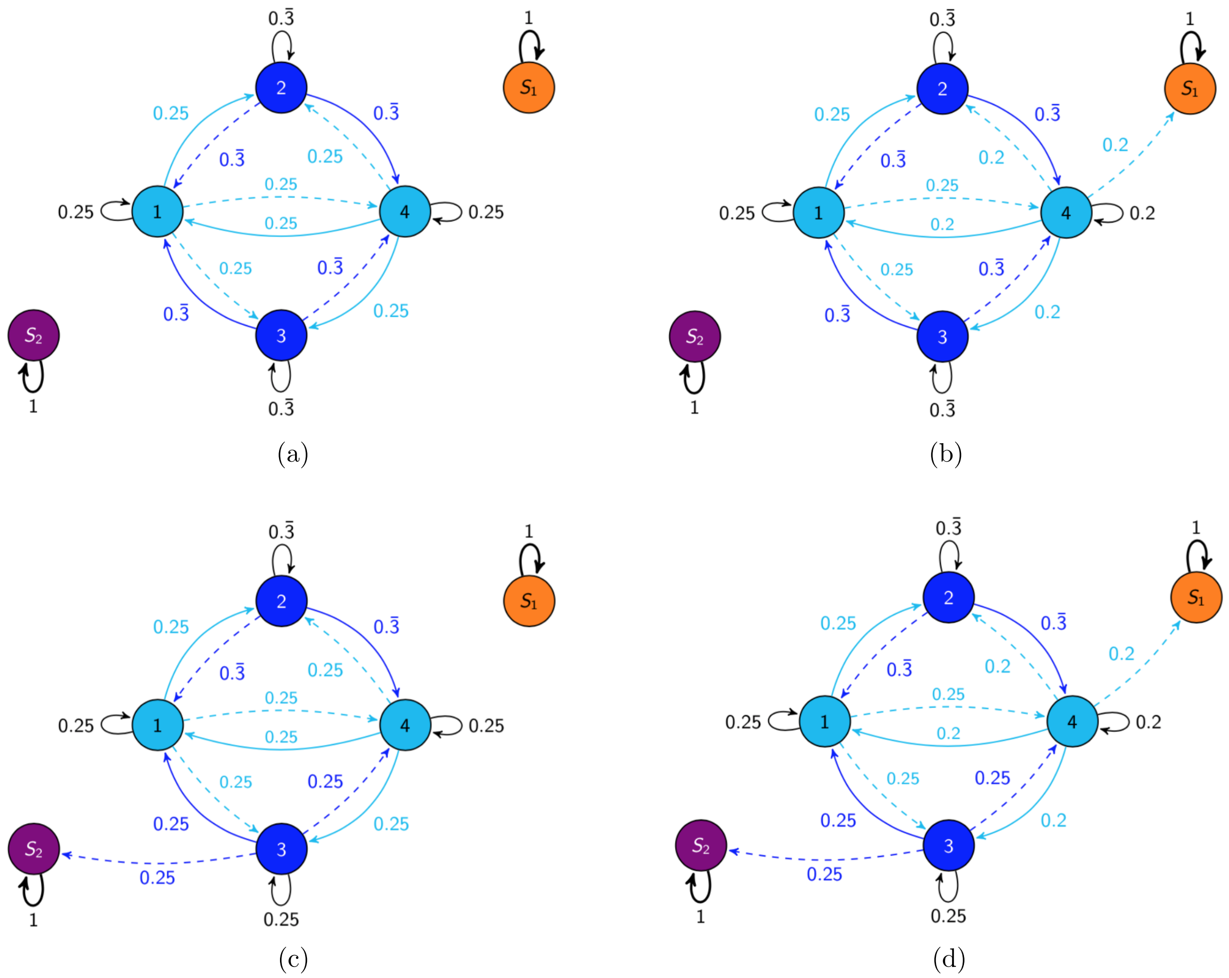


Figure 3: Networks composed by agents who recall and care about their past signals, where the intensity of each link of i is $\frac{1}{|\mathcal{N}_i|}$. $\theta^* = 10$, $\theta_{s1} = 15$, $\theta_{s2} = 5$.

N in all 4 cases. It is evident the effect of the constant flow of information, with only one stubborn the society does not converge toward the stubborn's opinion, while with two stubborn the average steady state opinion $\boldsymbol{\mu}'\mathbf{1}/n$ is closer to the truth than in a standard DeGroot updating model, as in *Example 1*. ▲

Corollary 1.1 tells us that, if agents are able to recall past signal their opinions are closer to the truth than with IR, nevertheless having PR of signals is not enough to have the convergence toward the truth. However, there exist particular situations in which a society with stubborn reaches the truth. It is trivial to see that all elements in $\boldsymbol{\mu}$ are equal to the truth θ^* if agents do not give importance to the stubborn, $\boldsymbol{a}_{s_s} = 0$ for all $s_s \in S$ or if the stubborns' opinion is equal to the truth itself, $\theta_{s_s} = \theta^*$. Moreover, if stubborn have the same influence over the society then the society can converge to the truth. For example with only two stubborn if $\boldsymbol{a}_{s_1} = \boldsymbol{a}_{s_2}$ and $\theta_{s_1} = 2\theta^* - \theta_{s_2}$ then $\boldsymbol{\mu} = \boldsymbol{\theta}^*$. Notice that in these cases the welfare is maximized both from the policy maker's and agents' prospective. I refer to the Appendix B for details and a deeper discussion about approaching the truth where there are stubborn agents in the society.

We can see that having more than one stubborn agent may facilitate learning if stubborn are evenly distributed around the truth. Therefore, from the point of view of a policy maker with the utility function $u_p(\boldsymbol{\mu}) := -(\boldsymbol{\mu} - \boldsymbol{\theta}^*)^2$, it may make sense to facilitate the entry of stubborn agents with different opinions, once there is already one in the network. This may suggest us that when the presence of a factious social media is recognized in the society, having more social media with different opinion may be desirable. Notice however that this argument is valid only if the network is exogenous, as in this paper. On the contrary, if agents have the possibility to choose their connection then, depending on the network structure and the strength of the signals, in the long-run stubborn can be isolated or different isolated communities where agents have opinion very close to the stubborn one may arise.

In next two sections (3.2 3.3), to convey results and, at the same time, to maintain formal simplicity, we consider a network with only one stubborn agent. We study, in 3.2, the distortion induced by a stubborn and, in 3.3, the optimal declaration of a sophisticated stubborn.

3.2 Marginal Distortion Induced by Stubborn

In order to better understand the effect of stubborn agents on the social learning, I analyze the effect of increasing (or decreasing) the influence of a stubborn agent s over a generic agent i of α_{is}.[22]

Let define $\hat{a}_{is} = a_{is} + \alpha_{is1}$ the new influence of a generic stubborn s on agent i and recall that the listening matrix should remain normalized (row-stochastic) then $\sum_{j \neq \{i,s\}} \hat{a}_{ij} = \sum_{j \neq \{i,s\}} \hat{a}_{ij} - \alpha_{is}$ ($\sum_{j \neq \{i,s\}} \alpha_{ij} = -\alpha_{is}$ and therefore $\alpha_{is} = -\sum_j \alpha_{ij}$). I further assume that a_{ii} remain fixed for all i, that $\alpha_{is} = \alpha$ and $\alpha_{ij} = \frac{\alpha}{n}$, for simplicity.

Let $\hat{\boldsymbol{\mathcal{A}}} := \boldsymbol{\mathcal{A}} - \mathbf{e}_i(\mathbf{g}_i)'$ be the modified matrix where $\mathbf{g}_i$ is the n-dimensional listening column vector that has: (i) 0 in its ith position, (ii) $\alpha/|\mathcal{N}_i|$ in all jth positions different from the one associated with the stubborn; $\mathbf{e}_i'$ is the is the n-dimensional row vector that has a 1 in its ith position and 0 elsewhere.

The next proposition describes the effect of introducing one stubborn agent in the so-

[22] Since in this section and in the next one we consider only one stubborn I call it s, instead of s_s.

ciety composed only by non-stubborn agents.

Proposition 2 *In a society without stubborn agents, if one agent create a link of intensity α with a stubborn, the marginal effect is*

$$\Delta\boldsymbol{\mu} = (\boldsymbol{C} - \boldsymbol{X})\alpha\boldsymbol{e}_i\theta_s - \boldsymbol{X}\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* \tag{10}$$

and the steady state opinion vector is

$$\hat{\boldsymbol{\mu}} = (\boldsymbol{C} - \boldsymbol{X})\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + \alpha\boldsymbol{e}_i\theta_s\right) \tag{11}$$

Moreover,

$$\boldsymbol{X} \cdot \mathbf{1} = \frac{\boldsymbol{C}\boldsymbol{e}_i(\boldsymbol{g}_i)'\boldsymbol{C}}{1 + (\boldsymbol{g}_i)'\boldsymbol{C}\boldsymbol{e}_i} \cdot \mathbf{1}$$

describe the distortion induced by the stubborn on agents' " updating centrality".

Proof. In the Appendix □.

The distortion induced by a stubborn in the opinion steady state vector is $|\Delta\boldsymbol{\mu}|$. From (10), it is evident that the distortion is increasing in the distance between θ_s and θ^*. Moreover, as shown by the distortion term $\boldsymbol{X}$, an increase (decrease) in stubborn influence creates a distortion in the centrality of all agents in the network, the new "updating centrality" is $(\boldsymbol{C} - \boldsymbol{X})$. [23]

If we consider a benevolent policy maker who want to minimize the distance between agents' opinion and the truth (with the previous utility function $u_p(\boldsymbol{\mu}) := -(\boldsymbol{\mu} - \boldsymbol{\theta}^*)^2$) the distortion $|\Delta\boldsymbol{\mu}|$ represents a measure of the policy maker's welfare loss. The welfare of policy maker is maximum when all the agents learn the truth, $|\Delta\boldsymbol{\mu}|$.

3.3 Monopolistic Stubborn Agent Problem

We have, until now, considered naive stubborn agents that always declare exactly the opinion that they want to disseminate. Let now consider a **sophisticated stubborn** that is able to optimally choose the opinion to declare in order to maximize the diffusion of the opinion that she really want to disseminate.

The stubborn knows the true state of the world θ^* and it is farsighted, namely she is able to compute the steady state opinion vector $\boldsymbol{\mu}$. The stubborn, given a fixed influence over agents, can declare any opinion $\theta_s^d \in \Theta$ such that minimize the distance between

[23] In this section, I study to a situation in which stubborn agents are not connected with the rest of the society ($\boldsymbol{a}_{s1} = \boldsymbol{a}_{s2} = \mathbf{0}$). In the Appendix I extend the results to generic $\boldsymbol{a}_{s1}$ and $\boldsymbol{a}_{s2}$.

agents' opinion and her own true opinion θ_s. I further assume that the stubborn faces a cost of lying (Kartik et al., 2007; Kartik, 2009). The cost of lying is assumed to be quadratic and proportional to difference between the true state of the world θ^* and the declared opinion θ_s^d, and is parameterized by $k \in \mathbb{R}_+$ which is the intensity of the lying cost.

The interpretation of the intensity of the lying cost, k, can be manifold, it can be thought as the punishment (fines) for having spread fake news, or as a cost to convince others about the reliability of your opinion (e.g. advertising cost), or can even represent the expected loss in credibility due to a too extreme declaration.

The stubborn choses to declare θ_s^d that solve the following problem

$$\max_{\theta_s^d} \; u_s(\boldsymbol{\mu}) := \; -(\boldsymbol{\mu} - \mathbf{1}\theta_s)^2 - k(\theta^* - \theta_s^d)^2 \tag{12}$$

Proposition 3 *Given, for all agents i, $p_{i,0}(\theta)$ the probability distribution over Θ at $t = 0$ (the prior), IR of past signals holds, and there is one sophisticated stubborn agent that solves the problem in (12), then at the steady state the probabilistic belief, $p_{i,\infty}(\theta)$, converges and the stubborn s would declare*

$$\theta_s^d = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta_s - \frac{\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1}\right)\right)'\boldsymbol{a}_s - k}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta^*. \tag{13}$$

Moreover, the steady state vector of opinions (beliefs) is

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G}) - \boldsymbol{a}_s\frac{\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1}\right)\right)'\boldsymbol{a}_s - k}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\right)\boldsymbol{\theta}^* + \boldsymbol{a}_s\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta_s\right) \tag{14}$$

Proof. In the Appendix □.

From (13), we can study conditions under which the stubborn declare a more extreme opinion than the one that she really has, $|\theta^* - \theta_s^d| > |\theta^* - \theta_s|$.

Without loss of generality we consider only the case where $\theta_s > \theta^* > 0$. From (13) we find $\bar{k} = \frac{(\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s - \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s)\theta_s - (\boldsymbol{C}(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1}))'\boldsymbol{a}_s\theta^*}{\theta_s - \theta^*}$ such that:

$$k < \bar{k} \quad \Leftrightarrow \quad \theta_s^d > \theta_s. \tag{15}$$

Whenever the cost of lying k overcomes the threshold $\bar{k}$, the sophisticated stubborn induces a lower opinions' distortion than a naive stubborn that declares exactly the opinion she wants to disseminate.

To better understand the implication of Proposition 3, I propose a numerical example.

Example 3: Let us consider the two societies described by Figure 3 (b) and Figure 3 (c). The stubborn in (b) and (c) want to disseminate $\theta^d_{s_1} = 15$ and $\theta^d_{s_2} = 5$, respectively. All agents have PR of past signals. The cost of lying for the stubborn is $k = 0.5$. In (b) the stubborn s_1 chooses $\theta^d_{s_1} = 17.22$ and the steady state average opinion is $\boldsymbol{\mu}'\mathbf{1}/n = 11.2897$. On the other hand, in (b) the stubborn s_2 chooses $\theta^d_{s_2} = 5.78$ and the steady state average opinion result to be $\boldsymbol{\mu}'\mathbf{1}/n = 8.2955$. Notice that the same cost of lying $k = 0.5$ is low enough for the more central stubborn s_1 to declare a more extreme opinion ($\theta^d_{s_1} > \theta s_1 > \theta^*$), and high enough for the less central s_2 to declare a less extreme opinion ($\theta^* > \theta^d_{s_2} > \theta s_1$). This suggest us that a more central stubborn has more room to spread misinformation in the society. ▲

Let define a society **smart** if the government can implement policy to enhance cost of lying ($k \uparrow$) or in which is the population's culture that, being is less tolerant to lies, have a higher cost of lying ($k \uparrow$), thus a society is more smart than another whenever its the cost of lying k is higher. The main message of this section is that in a smart society a naive stubborn – who, given her inability to make a declaration that maximizes her utility, declares her true opinion– is more dangerous than a sophisticated one. In fact, a naive stubborn pursuits is own agenda, to disseminate a certain opinion $\theta_s \neq \theta^*$, no matter on the cost that she faces. On the other hand, a sophisticated stubborn takes into account how costly is to declare a certain opinion. Therefore, the more the society is "smart" the more ineffective is the action of the sophisticated stubborn.

These results can provide us with insight about different political campaign strategies and allows us to explain, to a certain extent, the big differences between political statements before and after voting.

I have previously discussed that a policy may have incentives to introduce stubborn in the society to contrast the spread of misinformation and facilitate learning. In Appendix B, I discuss also the competition and the optimal declaration strategy of a stubborn controlled by policy maker to contrast the effect of a sophisticated stubborn in the society.

4 Temporary Misinformation

In previous sections, we have analyzed social learning only when there are permanent sources of misinformation (stubborn agents), in the society. In this section, I address the study of the speed of convergence to the consensus and the speed of learning of (5) in a network without stubborn agents but with temporary misinformation.

In the real world, there exist temporary sources of misinformation (e.g. rumors or fake news) that do not affect the long-run learning but may have important short-run effects. Gratton et al. (2017), for example, show that a bad sender (i.e. the spreader of misinformation in our setting) releases information later than good seder, this is mainly due to the fact that in the long-run fake news are unmasked.

Let us consider, for example, a massive diffusion of fake news, or misinformation, that foreruns an election. In such a case, if the learning take place at a too slow pace, the temporary distortion can seriously affect the election's outcome. Other examples where the speed of learning play a crucial role are the diffusion of misinformation regarding health, or climate change, issues where the longer the learning process the more serious the damages are. It is important to stress that in this cases the convergence to the consensus is not enough, in fact, agents may agree but still be far from the truth.

With updating rule as in (5), if the network is strongly connected (no stubborn), all agents learn the truth, in the long-run (Jadbabaie et al., 2012). However, in the short-run the speed of convergence can play a crucial role. Let us consider the steady state opinions' vector without any stubborn, $\boldsymbol{\mu} = \boldsymbol{\theta^*}$, and let assume that each agent i receive a shock ε_i that represents the diffusion of temporary fake news. The learning process start again by

$$\boldsymbol{\mu}_t = \boldsymbol{\theta^*} + \boldsymbol{\varepsilon}. \tag{16}$$

Since Jadbabaie et al. (2013) studies the role of different information among agents, in this paper I focus on the role of the network structure. For this reason I discuss the problem considering the case in which all agents are able to perfect recall their past signals, thus after some periods agents private information are equivalent.[24] Under PR the updating rule is

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{A}\boldsymbol{\mu}_t + \boldsymbol{D}\boldsymbol{\theta^*}. \tag{17}$$

To study the consensus time we avoid to consider the effect of the true state of the world θ^* on agents' updating. Using (16), (17) becomes

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{A}\boldsymbol{\mu}_t + \boldsymbol{D}(\boldsymbol{\mu}_t + \boldsymbol{\varepsilon}) = \boldsymbol{\mathcal{A}}\boldsymbol{\mu}_t + \boldsymbol{D}\boldsymbol{\varepsilon},$$

where $\boldsymbol{\varepsilon}$ represents a small deviation from the steady state. There is consensus when agents' opinion are very similar to each other, regardless of the initial shock. Therefore $\boldsymbol{D}\boldsymbol{\varepsilon}$ is negligible and the consensus time depends only on the convergence of $\boldsymbol{\mathcal{A}}$. Thus,

[24]Since we assume a shock at the steady state, under PR all agents have collect severals signals therefore the first moment of their bayesian posterior is exactly θ^*.

the consensus dynamics of (17) is equivalent to

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mathcal{A}}\boldsymbol{\mu}_t. \tag{18}$$

Therefore we can use the standard definition of consensus used by Golub and Jackson (2012). Using the standard l^2-norm, I define consensus time as the time it takes for the distance, between average sum of current opinions and steady state opinions of (18), to get below an arbitrary small enough ϵ:

Definition [Consensus Time] *The consensus time to $\epsilon > 0$ of a connected graph $\mathcal{G}$ is*

$$CT(\epsilon, \mathcal{G}) := \sup_{\mu_t \in \mathbb{R}^n} \min\{T : ||\boldsymbol{\mathcal{A}}^T \boldsymbol{\mu}_t - \boldsymbol{\mathcal{A}}^\infty \boldsymbol{\mu}_t|| < \epsilon\}.$$

Taking the supremum allows us to consider the worst case as a benchmark.

To define the learning time, I still use the l^2-norm. The learning time is the time T, such that the distance between the opinion vector $\boldsymbol{\mu}_{t+T}$ and the true state of the world $\boldsymbol{\theta}^*$, is less then ϵ:

Definition [Learning Time] *The learning time to $\epsilon > 0$ of a connected graph $\mathcal{G}$ is*

$$LT(\epsilon, \mathcal{G}) := \sup_{\mu_t \in \mathbb{R}^n} \min\{T : ||\boldsymbol{\mu}_{t+T} - \boldsymbol{\theta}^*|| < \epsilon\}.$$

Again, I use the supremum to consider the worst case as a benchmark.

Notice that in our problem the learning time can be different from the consensus time. Agents may have very similar opinions but still be far from the truth and, receiving new pieces of information at each period, they approach $\boldsymbol{\theta}^*$ until they reach it.[25]

To understand how the network topology affect the consensus and the learning time, it is now important to define a "bottleneckedness" measure of a network.

Definition [Cheeger Constant] *The Cheeger Constant of the graph $\mathcal{G}(N, \boldsymbol{\mathcal{A}})$ is*

$$\phi(\mathcal{G}) = \min_{S \subseteq N} \sum_{i \in S} \sum_{j \notin S} \frac{a_{ij}}{|S||S^c|}.$$

where $S \cup S^c = N$.

The Cheeger constant is a measure of whether or not a graph has a "bottleneck". It quantifies how the network $\mathcal{G}$ can be partitioned in two components. If $\phi(\mathcal{G})$ is small then

[25]For example, in the past many people did not believe that smoking cigarette was harmful to health.

the network is composed by two sets of vertices with few links between them. On the other hand, if $\phi(\mathcal{G})$ is large, then the network has many links between those two subsets. Moreover, the Cheeger constant is strictly positive if and only if the network is connected.[26]

Defining $1 = \lambda_1^{\boldsymbol{\mathcal{A}}} \geq \lambda_2^{\boldsymbol{\mathcal{A}}} \geq \lambda_n^{\boldsymbol{\mathcal{A}}}$ the eigenvalues of the matrix $\boldsymbol{\mathcal{A}}$.

Proposition 4 *Given the updating rule* (17) *and a network represented by the adjacency matrix* $\boldsymbol{\mathcal{A}}$*, then for any* $\epsilon > 0$ *the consensus time* $CT(\epsilon, \mathcal{G})$ *is in the order of* $\lambda_2^{\boldsymbol{\mathcal{A}}}$ *exponentially. Moreover,*

$$\frac{\phi(\mathcal{G})^2}{2} \leq 1 - \lambda_2^{\boldsymbol{\mathcal{A}}} \leq 2\phi(\mathcal{G}). \tag{19}$$

Proof. in the Appendix. □

This result is standard and consistent with previous literature. In fact, the matrix $\boldsymbol{\mathcal{A}}$ is stochastic and we know from the theory of Markov chains that speed of convergence negatively depends on the magnitude of the second eigenvalue of $\boldsymbol{\mathcal{A}}$. Moreover, if shocks are correlated, $\phi(\mathcal{G})$ is strictly related with the *spectral homophily* measure of Golub and Jackson (2012). The main idea is that the speed of convergence depends on the second largest eigenvalue of $\boldsymbol{\mathcal{A}}$, and according to Cheeger's inequality the second smallest eigenvalue of the Laplacian matrix $\lambda_{n-2}^{\mathcal{L}_{\boldsymbol{\mathcal{A}}}}$ is an approximation of the Cheeger constant.[27] We can prove that, in this model, $\lambda_2^{\mathcal{L}_{\boldsymbol{\mathcal{A}}}} = 1 - \lambda_{n-2}^{\boldsymbol{\mathcal{A}}}$, therefore the smaller $\lambda_2^{\boldsymbol{\mathcal{A}}}$ is, the faster the consensus occurs and the more connected the two subsets of nodes are.[28]

Figure 4 clearly shows how the more the two subsets of nodes are connected, the faster the opinion to converge toward the consensus. In this example, lower "bottleneckedness" means higher homophily, in fact, shocks in $\boldsymbol{\varepsilon}$ are of opposite sign for agents belonging to the two subgroups. We can further see, that a faster consensus does not translate in a faster learning. In these examples (Figure 4) it is possible to see see that, the society with more "bottleneckedness" (less homophily) is the first to learn the truth in average. Therefore here the learning time does not depend only on the "bottleneckedness" (or homophily) of the network as in Golub and Jackson (2012).

To study the speed of learning the truth, we go back to consider the dynamics in (17) where the role of θ^* is explicit. Let recall that the steady state of (17) is $\boldsymbol{\theta}^*$ and that $\boldsymbol{A}$ is a sub stochastic matrix, in fact $\boldsymbol{\mathcal{A}} = \boldsymbol{A} + \boldsymbol{D}$.

[26]The first formulation of Cheeger constant is due to Cheeger (1969), for a deeper discussion of discrete version I refer to Chapter 2 and 6 of Chung (1996).

[27]In the appendix I provide technical details and formal definition of Chegeer's inequality and Laplacian matrix. For a deeper discussion about Chegeer's inequality I still refer to Chapter 2 and 6 of Chung (1996).

[28]The second-smallest eigenvalue of the Laplacian matrix $\lambda_2^{\mathcal{L}_{\boldsymbol{\mathcal{A}}}}$ is known as the algebraic connectivity of the graph, and is greater than 0 if and only if the graph described by the adjacency matrix $\boldsymbol{\mathcal{A}}$ is a connected graph.

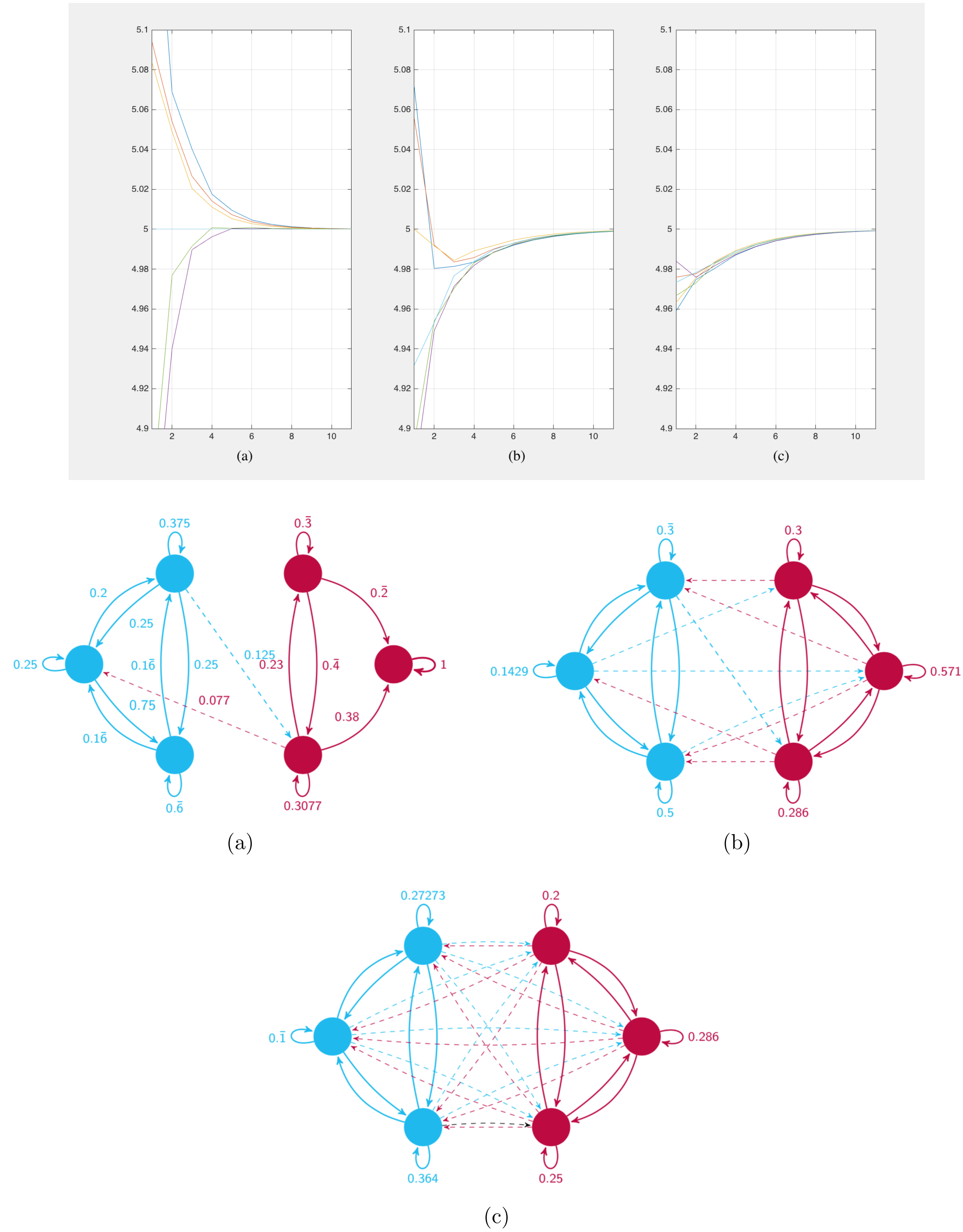


Figure 4: Convergence of $\boldsymbol{\mu}_t$ to $\theta^* = 5$ in networks with, $\varepsilon_i = 0.25$ for $i = 1, 2, 3$ and $\varepsilon_i = -0.25$ for $i = 4, 5, 6$. (a) $\lambda_2^{\boldsymbol{A}} = 0.9760$, (b) $\lambda_2^{\boldsymbol{A}} = 0.5991$, (c) $\lambda_2^{\boldsymbol{A}} = 0.2283$

Let us define $\bar{a}_{ii} := \sum_i^{n-1} \frac{a_{ii}}{n-1}$ and ${}_{\min}a_{ii}$ as the average and the minimum among self-weights, respectively. And $\kappa(U) := ||\boldsymbol{U}||||\boldsymbol{U}^{-1}||$ as the condition number of the eigenvector basis U.

Proposition 5 *Given the updating rule* (17) *and a network represented by the adjacency matrix $\boldsymbol{\mathcal{A}}$, then for any $\epsilon > 0$*

$$LT(\epsilon, \mathcal{G}) \leq \lceil \frac{log(\epsilon/(\kappa(U))}{log(|\lambda_1^{\boldsymbol{A}}|)} \rceil \quad (20)$$

Moreover,

$$\min_i \{a_{ii}\} \leq 1 - \lambda_1^{\boldsymbol{A}} \leq \bar{a}_{ii} \quad (21)$$

Proof. in the Appendix. $\square$

The first part of Proposition 5 shows that, the $LT(\epsilon, \mathcal{G})$ is of the order of the higher eigenvalue of the substochastic matrix $\boldsymbol{A}$ which represent the network without self-weights, and not on the eigenvalue of the full adjacency matrix $\boldsymbol{\mathcal{A}}$. Moreover, positively depends also on the condition number of the eigenvector basis. The second part of Proposition 5 shed light on the role of self-reliance for the speed of learning in our problem. The minimum self-loop $\min_i\{a_{ii}\}$ – which represents the "weakest link" of the learning process, namely the agent that gives less weight to her own private information— is related, through $\lambda_1^{\boldsymbol{A}}$, to the lower bound of the time of learning. Namely, the higher the minimum individual consideration about the stream of information is, then the faster the learning can be, at its minimum. Thus, it is extremely important how bad informed is the less informed agent, improving the lower level it is possible to ensure a minimum speed of learning. On the other hand, the average self-loop $\bar{a}_{ii}$ – that is the attention that the society gives, on average, to her own private information– can provide information about the upper bound of speed of learning, namely the higher the average individual consideration about the stream of information is, then the faster the learning can be, at its maximum.

Notice that if $a_{ii} = a_{jj} = \alpha$ for all $i, j \in N$ then, from (21), $\alpha = 1 - \lambda_1^{\boldsymbol{A}}$. Moreover from (20) $LT(\epsilon, \mathcal{G}) \leq \lceil \frac{log(\epsilon/(\kappa(U))}{log(1-\alpha)} \rceil$.

We have seen, in Proposition 4 and 5, how the "bottleneckedness" (homophily) of the network plays a crucial role in the consensus time, while self-weights are fundamental for what concerns the learning time.

It is important to stress that the learning time, even if ultimately depends on $\lambda_1^{\boldsymbol{\mathcal{A}}}$, in the short-run is affected even by other eigenvalues, depending on the magnitude.[29] As an

[29] I refer to the proof of Proposition 5 for details.

approximation I believe that is enough to consider only $\lambda_1^{\mathcal{A}}, \lambda_2^{\mathcal{A}}$, the two largest eigenvalues.[30] The main intuition is that given two societies with similar levels of self-weights, different learning times are due to the difference on the second largest eigenvalues. Unfortunately, in this case, the interpretation of $\lambda_2^{\mathcal{A}}$ as "bottleneckedness" of the networks is not straightforward.

Corollary 5.1 *Let us consider two symmetric graphs* $\mathcal{G}_1(N, \boldsymbol{\mathcal{A}}_1)$ *and* $\mathcal{G}_2(N, \boldsymbol{\mathcal{A}}_2)$*, where* $\boldsymbol{\mathcal{A}}_1 = \alpha \boldsymbol{I} + \boldsymbol{A}_1$ *and* $\boldsymbol{\mathcal{A}}_2 = \alpha \boldsymbol{I} + \boldsymbol{A}_2$*, with* $\alpha \in [0, 1]$*. Given the updating rule* (17)*, if* $\phi(\mathcal{G}_1) \geq \phi(\boldsymbol{G}_2)$ *then for any* $\epsilon > 0$

- $CT(\epsilon, \mathcal{G}_1) \leq CT(\epsilon, \mathcal{G}_2)$
- $LT(\epsilon, \mathcal{G}_1) \leq LT(\epsilon, \mathcal{G}_2)$

Proof. in the Appendix □

Corollary 5.1 discusses the particular case where all agents in two different symmetric networks give the same weight to the information they receive. In this case the ordering of second largest eigenvalues of adjacency matrices relative to the network without self-loops ($\lambda_2^{\boldsymbol{\mathcal{A}}_1}$ and $\lambda_2^{\boldsymbol{\mathcal{A}}_2}$) is the same as the ordering of "bottleneckedness" measures ($\phi(\mathcal{G}_1)$ and $\phi(\mathcal{G}_2)$). In Figure 5, we can observe a numerical example where all agents give the same weights to their private information and as a result a higher homophily leads to a faster consensus and learning.

From Proposition 4 and 5 and Corollary 5.1 we deduce that, if agents receive a continuous stream of new information about the true state of the world then the speed of learning and the speed of convergence to the consensus are, in general, different. The first one mainly depends on the self-reliance of agents, namely how much they care about the information that they receive. The second strongly depends on the "bottleneckedness" of the network and therefore, when shocks are correlated with network structure, on the homophily. If all agents have the same fixed reliance on private signals and the network is symmetric then a smaller "bottleneckedness" (higher homophily) leads both to a lower consensus and learning time.

Comparing results with Golub and Jackson (2012) the main insight is that, if agents receive new information at each time then the learning time is not necessarily directly correlated to homophily. For example, an increase in the number of connections among agents belonging to two different subset of the network[31] may translate into a higher LT

[30]Notice, in fact, that $\boldsymbol{\mathcal{A}}$ is row sub-stochastic therefore all the eigenvalues are less then 1 and most of them vanish after very few iterations.

[31]That, as said, correspond to a decrease in the level of homophily if shocks are correlated.

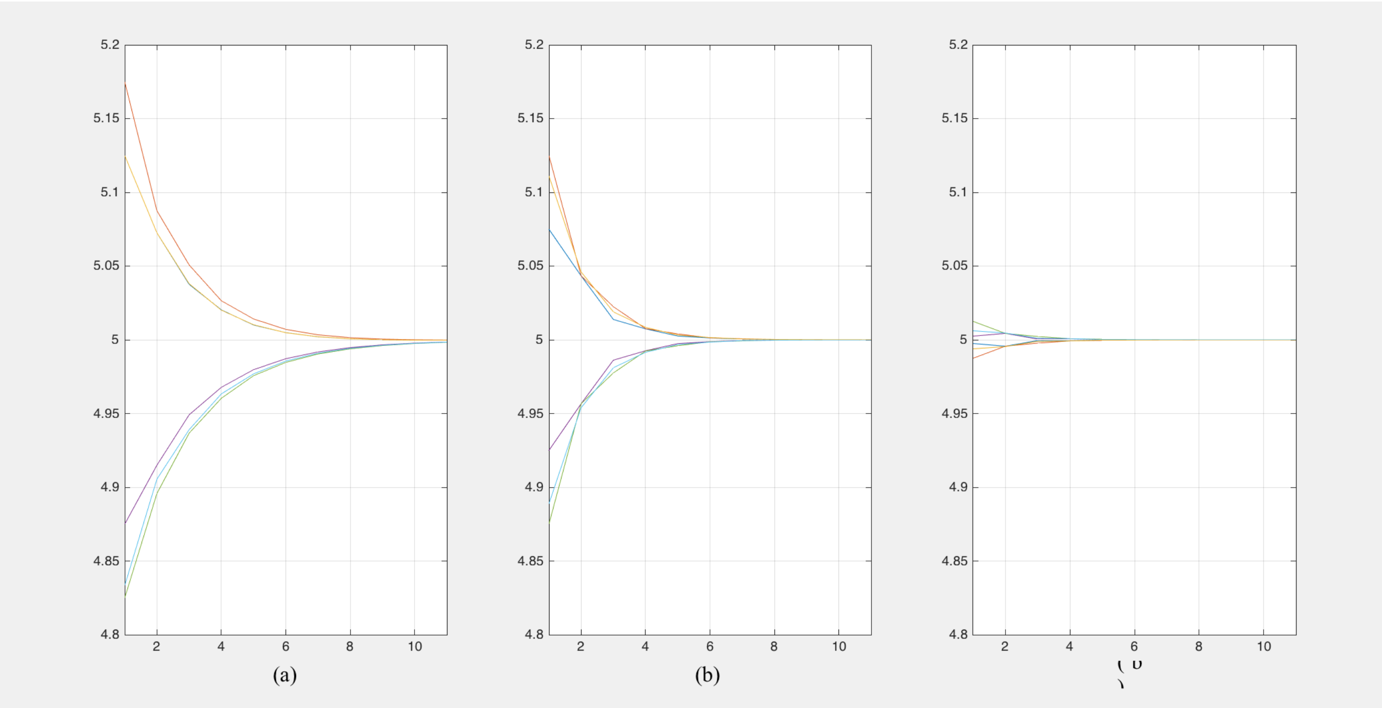


Figure 5: Convergence of $\boldsymbol{\mu}_t$ to $\theta^* = 5$ in networks with equal self-loops $\alpha = 0.3$, $\theta^* = 5$, $\varepsilon_i = 0.25$ for $i = 1, 2, 3$ and $\varepsilon_i = -0.25$ for $i = 4, 5, 6$. (a) $\lambda_2^{\boldsymbol{A}} = 0.5846$, (b) $\lambda_2^{\boldsymbol{A}} = 0.4209$, (c) $\lambda_2^{\boldsymbol{A}} = -0.1$

(as in Figure 4), due to the rescaling of self-weights.[32]

From a policymaker point of view, these results suggest an increase in the network density does not directly translate into a faster learning of the truth. Moreover, policies that aim to increase social interaction and decreasing the homophily to facilitate agents' learning, may succeed only if agents belonging to different groups have very similar levels of "good information". On the contrary, the learning of better informed agents is slowed down by others and this reduce the speed of learning of the whole society.

5 Conclusion

This paper addresses the problem of the spread of misinformation in a social network where agents interact to learn an underlying state of the world with a non-Bayesian social learning process. The main difference with the standard naive social learning is the continuous stream of new signals that agents receive at each period. This implies a stronger connection to the truth in the learning process. Considering the permanent misinformation or opinion manipulation pursuit by "prominent" agents the paper shows that despite

[32]Given that $\sum_j a_{ij} = 1$ for all $j \in I$, if for i a_{ik} increase, where $k \in I/\{i\}$, then the sum of others a_{ij}, for $j \in I/\{k\}$ should decrease of the same amount, thus a_{ii} can decrease too.

receiving new signals every period, agents are not able to learn the underlying state of the word nor to reach a consensus. This depends on the fact that the network is not strongly connected due to the presence of "prominent agents" who behave as stubborn. Differently from the benchmark of DeGroot model, if agents receive signals at each period, the steady state agents' opinion does not depends only on the stubborn opinion and the centrality of agents connected to stubborn but also on the true state of the world and on agents' self-reliance. The paper also introduce a novel centrality measure the "updating centrality" that, in the case of perfect recall of past signals, corresponds to the Katz-Bonachic centrality. I further characterize the optimal action of a stubborn who want to manipulate the opinion dynamics showing his relationship with the cost of disseminating misinformation. At the end, we discuss the consensus and learning time after an exogenous shock that temporarily moves opinions away from the steady state. I prove that the speed of reaching the consensus inversely depends on the "bottleneckedness", and thus the homophily, of the underlying network, while the speed of convergence toward the truth mainly depends on the strength of self-weights.

A potential interesting extension for the future is to study a similar framework in a society where agents have the possibility to choose their connection. The probable result is that if stubborn agents are not very central and agents strongly rely on their private signals, then stubborn agents end up to be isolated in the long-run and agents reach a full learning. On the other hand, if stubborn is central and agents have low reliance on private signals then different isolated communities where agents have opinion very close to the stubborn can arise. In this framework, it is interesting to investigate if a sophisticated stubborn would declare more extreme opinions when she is central or peripheral. This analysis can be useful to investigate the role of different information structure in the arising of polarization phenomena.

A Appendix

A.1 Lemma 1

Before presenting the proof of Proposition 1 and Corollary 1.1 we have to state and proof the following lemma.[33]

Lemma 1 (Jadbabaie et al., 2012) *Let $\boldsymbol{\mathcal{A}}_s$ denote the matrix of social interactions. The sequence $\sum_{i=1}^{n} \nu_i p_{i,t}(\theta^*)$ converges $\mathbb{P}^*$ - almost surely as $t \to \infty$, where $\boldsymbol{\nu}$ is any non-negative left eigenvector of $\boldsymbol{\mathcal{A}}_s$ corresponding to its unit eigenvalue.*

Proof.
Notice that since $\boldsymbol{\mathcal{A}}_s$ is stochastic its largest eigenvalue is $\lambda_1 = 1$. Moreover, always exists $\boldsymbol{\nu}$ a non-negative left eigenvector corresponding to the eigenvalue $\lambda_1 = 1$. We define $\boldsymbol{p}_{t+1}(\theta) := (p_{i,t+1}(\theta))_{i \in N}$ the vector of probabilistic belief of all $i \in N$ and $p_{s1,t+1}(\theta), p_{s2,t+1}(\theta)$ the stubborn beliefs. Since stubborn do not revise their beliefs we can consider only the probabilistic belief's updating rule (1) for all $i \in N$ (evaluated at θ^*)

$$\boldsymbol{p}_{t+1}(\theta^*) = \boldsymbol{\mathcal{A}} \boldsymbol{p}_t(\theta^*) + \sum_{i=1}^{n} p_{i,t}(\theta^*) a_{ii} \left(\frac{l(\omega_{i,t+1}|\theta^*)}{\int_\Theta l(\omega_{i,t+1}\theta^*) p_{i,t}(\theta^*) d\theta} - 1 \right) + \sum_{i=1}^{n} \left[a_{i,s_1} p_{s_1}(\theta^*) + a_{i,s_2} p_{s_2}(\theta^*) \right] \tag{22}$$

$\boldsymbol{\mathcal{A}}$ is a stochastic matrix, therefore the largest eigenvalue is $\lambda_1 = 1$, we denote with $\boldsymbol{\nu}$ the eigenvector corresponding to λ_1. Notice that all element in $\boldsymbol{\nu}$ are non negative and $\boldsymbol{\nu}' \boldsymbol{\mathcal{A}} = \boldsymbol{\nu}' \lambda_1$. Moreover, for a generic stubborn $p_s(\theta) = 0$ for all $\theta \neq \theta_s$.

Let us multiply both sides of (20) by $\boldsymbol{\nu}'$,

$$\boldsymbol{\nu}' \boldsymbol{p}_{t+1}(\theta^*) = \boldsymbol{\nu}' \boldsymbol{\mathcal{A}} \boldsymbol{p}_t(\theta^*) + \sum_{i=1}^{n} \nu_i p_{i,t}(\theta^*) a_{ii} \left(\frac{l(\omega_{i,t+1}|\theta^*)}{\int_\Theta l(\omega_{i,t+1}\theta^*) p_{i,t}(\theta^*) d\theta} - 1 \right) + \underbrace{\sum_{i=1}^{n} \nu_i \left[a_{i,s_1} p_{s_1}(\theta^*) + a_{i,s_2} p_{s_2}(\theta^*) \right]}_{=0}$$

then we take the expectation $\mathbb{E}$ associated with measure $\mathbb{P}^*$ with respect the filtration $\mathcal{F}_t$

$$\mathbb{E} \left[\boldsymbol{\nu}' \boldsymbol{p}_{t+1}(\theta^*) | \mathcal{F}_t \right] = \boldsymbol{\nu}' \boldsymbol{p}_t(\theta^*) + \sum_{i=1}^{n} \nu_i p_{i,t}(\theta^*) a_{ii} \mathbb{E} \left[\left(\frac{l(\omega_{i,t+1}|\theta^*)}{\int_\Theta l(\omega_{i,t+1}\theta^*) p_{i,t}(\theta^*) d\theta} - 1 \right) | \mathcal{F}_t \right]$$

[33] We adapt the proof from Jadbabaie et al. (2012), the only difference is the presence of stubborn agents that makes $\boldsymbol{\mathcal{A}}_s$ not to be strongly-connected.

Jensen's inequality implies that

$$\mathbb{E}\left[\left(\frac{l(\omega_{i,t+1}|\theta^*)}{\int_\Theta l(\omega_{i,t+1}\theta^*)p_{i,t}(\theta^*)d\theta}\right)|\mathcal{F}_t\right] \geq \left(\mathbb{E}\left[\left(\frac{l(\omega_{i,t+1}|\theta^*)}{\int_\Theta l(\omega_{i,t+1}\theta^*)p_{i,t}(\theta^*)d\theta}\right)^{-1}|\mathcal{F}_t\right]\right)^{-1} = 1$$

then

$$\mathbb{E}\left[\boldsymbol{\nu}'\boldsymbol{p}_{t+1}(\theta^*)|\mathcal{F}_t\right] = \boldsymbol{\nu}'\boldsymbol{p}_t(\theta^*) + \sum_{i=1}^{n} \nu_i p_{i,t}(\theta^*) a_{ii} \underbrace{\mathbb{E}\left[\left(\frac{l(\omega_{i,t+1}|\theta^*)}{\int_\Theta l(\omega_{i,t+1}\theta^*)p_{i,t}(\theta^*)d\theta} - 1\right)|\mathcal{F}_t\right]}_{\geq 0}$$

Therefore

$$\mathbb{E}\left[\boldsymbol{\nu}'\boldsymbol{p}_{t+1}(\theta^*)|\mathcal{F}_t\right] \geq \boldsymbol{\nu}'\boldsymbol{p}_t(\theta^*)$$

Thus, since $\boldsymbol{\nu}'\boldsymbol{p}_t(\theta^*)$ is a submartingale with respect $\mathcal{F}_t$ then it converges $\mathbb{P}^*$ - almost surely.

□

A.2 Proof of Proposition 1

Signals are drawn from a Gaussian distribution. Being $p_{i,0}(\theta)$ normally distributed too, it is a conjugate prior for all $i \in N$. Thus the mean of the bayesian posterior distribution is a convex combination of the mean of the prior and the received signal. Moreover, the higher the precision of the prior the lower the strength of signal on the posterior, and its average. In particular

$$\bar{\beta}_{i,1} = \int_\Theta \theta\beta_{i,1}d\theta = \gamma^n_{i,0}\mu_{i,t} + (1-\gamma^n_{i,0})\omega_{i,1} \tag{23}$$

where $\omega_{i,1}$ is the signal received by agent i at 1 and $\gamma^n_{i,0} = \frac{\tau^p_{i,0}}{\tau^p_{i,0}+\tau^\omega}$ (the superscript n stands for normal, in fact it depends on the normality assumption). At each period agents according to (4) update their probabilistic belief trough a convex combination of their bayesian posterior and their neighborhoods' probabilistic beliefs, thus $p_{i,t}(\theta)$ with $t > 1$ is always a mixture of Gaussians. From Lemma 3.4.2 of Robert (2007), we know that given a natural conjugate family of an exponential family, then the set of mixtures of n conjugate distributions, is also a conjugate family and the posterior distribution is a mixture of the posterior of each component of the mixture. For this reason the first moment of the posterior is always a convex combination of convex combinations between the prior mean and the received signal, moving on with time the prior is a convex combination of the convex combination at the previous period and so on. Therefore, at a generic time $t > 1$

$$\bar{\beta}_{i,t+1} = \int_\Theta \theta\beta_{i,t+1}d\theta = \gamma_{i,t}\mu_{i,t} + (1-\gamma_{i,t})\omega_{i,t+1}, \tag{24}$$

where $\gamma_{i,t}$ depends also on the weight of the mixture (elements of a_{ij} for all $j \in N$) and it is increasing in $\tau^p_{i,t}$. Thus (4) becomes

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{D}(\boldsymbol{G}_t\boldsymbol{\mu}_t + (\boldsymbol{I} - \boldsymbol{G}_t)\boldsymbol{\omega}_t) + \boldsymbol{A}\boldsymbol{\mu}_t + \sum_{s_s \in S} \boldsymbol{a}_{s_s}\theta_{s_s} \tag{25}$$

From Lemma 1, we know that the probability distribution converges, $p_{i,t}(\theta) \to p_i(\theta) := p_{i,\infty}(\theta)$ therefore both the mean and the precision always converges almost surely $\mu_{i,t} \to \mu_i := \mu_{i,\infty}$, $\tau^p_{i,t} \to \tau^p_i := \tau^p_{i,\infty}$ and thus $\gamma_{i,t} \to \gamma_i := \gamma_{i,\infty}$. Moreover, we recall that $E[\boldsymbol{\omega}] = \boldsymbol{\theta^*}$. Thus, by the law of large numbers at the steady state

$$\boldsymbol{\mu} = \boldsymbol{D}(\boldsymbol{G}\boldsymbol{\mu} + (\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta^*}) + \boldsymbol{A}\boldsymbol{\mu} + \sum_{s_s \in S} \boldsymbol{a}_{s_s}\theta_{s_s}$$

That lead us to

$$\Rightarrow \boldsymbol{\mu} = \underbrace{(\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1}}_{\boldsymbol{C}} \left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta^*} + \sum_{s_s \in S} \boldsymbol{a}_{s_s}\theta_{s_s} \right)$$

□

A.3 Proof of Corollary 1.1

If agents are able to recall all the past signals after T period they will compute their bayesian posterior using all the T signals. We know that the more data we have the more the data overwhelm the prior and dominate the Bayesian posterior. Therefore as $T \to \infty$,

$$\lim_{T \to \infty} \gamma_{i,T} = 0$$

Then all elements of matrix $\boldsymbol{G}$ approach zero time to time. At the steady state

$$\Rightarrow \boldsymbol{\mu} = (\boldsymbol{I} - \boldsymbol{A})^{-1} \left(\boldsymbol{D}\boldsymbol{\theta^*} + \sum_{s_s \in S} \boldsymbol{a}_{s_s}\theta_{s_s} \right)$$

□

A.4 Proof of Proposition 2

In order to prove Proposition 2 we state this well known linear algebra result.

Sherman-Morrison Formula. (Sherman and Morrison, 1950) *Let $\boldsymbol{B}$ be a nonsingular n-dimensional real matrix, and $\boldsymbol{u}, \boldsymbol{v}$ two real n-dimensional column vectors such that $1 + \boldsymbol{v}' A^{-1} \boldsymbol{u} \neq 0$. Then,*

$$(\boldsymbol{B} + \mathbf{u}\mathbf{v}')^{-1} = \boldsymbol{B}^{-1} - \frac{\boldsymbol{B}^{-1}\mathbf{u}\mathbf{v}'\boldsymbol{B}^{-1}}{1 + \mathbf{v}'\boldsymbol{B}^{-1}\mathbf{u}}$$

Since $\boldsymbol{G}$ is a diagonal matrix and α_{ii} is assumed to be zero, then we can apply the Sherman-Morrison Formula, where $\boldsymbol{B} = (\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})$, $\mathbf{u} = \mathbf{e}_i$ and $\mathbf{v} = \mathbf{g}_i$.

We study the general case. Let us consider the steady state opinion vector.

$$\boldsymbol{\mu} = (\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1} (\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + \boldsymbol{a}_{s1}\theta_{s1} + \boldsymbol{a}_{s2}\theta_{s2})$$

If agents i increase the interaction with the stubborn s_1 of α then will decrease proportionally the interaction of other agents as captured by the vector $\boldsymbol{g}_i$. The new interaction matrix is $\hat{\boldsymbol{\mathcal{A}}} := \boldsymbol{\mathcal{A}} - \mathbf{e}_i\mathbf{g}_i'$. Since $\mathbf{e}_i(\mathbf{g}_i)'$ does not affect the main diagonal $\boldsymbol{D}$, the new opinion vector. $\hat{\boldsymbol{\mu}}$ is

$$\Rightarrow \hat{\boldsymbol{\mu}} = (\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A} + \mathbf{e}_i\mathbf{g}_i')^{-1} (\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + (\boldsymbol{a}_{s1} + \alpha\mathbf{e}_i)\theta_{s1} + \boldsymbol{a}_{s2}\theta_{s2})$$

By Sherman-Morrison Formula we know that

$$(\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A} + \mathbf{e}_i\mathbf{g}_i')^{-1} = (\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1} - \frac{(\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1}\mathbf{e}_i\mathbf{g}_i'(\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1}}{1 + \mathbf{g}_i'(\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1}\mathbf{e}_i}$$

The first term on the right side is $\boldsymbol{C}$. We name $\boldsymbol{X}$ the second one. Thus we obtain

$$\Rightarrow \hat{\boldsymbol{\mu}} = (\boldsymbol{C} - \boldsymbol{X}) (\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + (\boldsymbol{a}_{s1} + \alpha\mathbf{e}_i)\theta_{s1} + \boldsymbol{a}_{s2}\theta_{s2})$$

$$\Rightarrow \hat{\boldsymbol{\mu}} = \underbrace{\boldsymbol{C} (\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + \boldsymbol{a}_{s1}\theta_{s1} + \boldsymbol{a}_{s2}\theta_{s2})}_{\boldsymbol{\mu}} + \underbrace{((\boldsymbol{C} - \boldsymbol{X})\alpha\mathbf{e}_i - \boldsymbol{X}\boldsymbol{a}_{s1})\theta_{s1} - \boldsymbol{X}(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + \boldsymbol{a}_{s2}\theta_{s2}))}_{\Delta\boldsymbol{\mu}}$$

If we want to see the effect of creating the first link of intensity α with a stubborn for a generic i it is enough to consider a interaction matrix where $\boldsymbol{a}_{s1} = \boldsymbol{a}_{s2} = 0$, thus

$$\hat{\boldsymbol{\mu}} = \underbrace{\boldsymbol{C} (\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^*)}_{\boldsymbol{\mu}} + \underbrace{(\boldsymbol{C} - \boldsymbol{X})\alpha\mathbf{e}_i\theta_{s1} - \boldsymbol{X}\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^*)}_{\Delta\boldsymbol{\mu}}$$

$$\Rightarrow \hat{\boldsymbol{\mu}} = (\boldsymbol{C} - \boldsymbol{X})(\alpha\mathbf{e}_i\theta_{s1} + \boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^*)$$

We can see that

$$X = \frac{(\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1}\mathbf{e}_i\mathbf{g}_i'(\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1}}{1 + \mathbf{g}_i'(\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1}\mathbf{e}_i}$$

measures the variation of the updating centrality after introducing a stubborn in the society.

□

A.5 Proof of Proposition 3

The convergence of the probability distribution is ensured by Lemma 1 and Proposition 1.

Let us consider the maximization problem described in (12)

$$\max_{\theta_s^d} \; u_s(\boldsymbol{\mu}) := \; -(\boldsymbol{\mu} - \mathbf{1}\theta_s)^2 - k(\theta^* - \theta_s^d)^2$$

Expanding the first term we obtain

$$\max_{\theta_s^d} \; -\boldsymbol{\mu}'\boldsymbol{\mu} + 2(\mathbf{1}\theta_s)'\boldsymbol{\mu} - (\mathbf{1}\theta_s)'(\mathbf{1}\theta_s) - k(\theta^* - \theta_s^d)^2$$

We can see that this problem is equivalent to

$$\max_{\theta_s^d} \; -\boldsymbol{\mu}'\boldsymbol{\mu} + 2(\mathbf{1}\theta_s)'\boldsymbol{\mu} - k(\theta^* - \theta_s^d)^2 \tag{26}$$

With only one sophisticated stubborn agent that declares θ_s^d the steady state opinion dynamics is

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + \boldsymbol{a}_s\theta_s^d\right) \tag{27}$$

Substituting (27) in (26) we obtain

$$\max_{\theta_s^d} \; -2\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^*\right)\right)'\boldsymbol{a}_s\theta_s^d - \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\theta_s^{d2} + 2(\mathbf{1}\theta_s)'\boldsymbol{C}\boldsymbol{a}_s\theta_s^d - k(\theta^* - \theta_s^d)^2$$

We solve the First Order Condition:

$$-2\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^*\right)\right)'\boldsymbol{a}_s - 2\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\theta_s^d + 2(\mathbf{1}\theta_s)'\boldsymbol{C}\boldsymbol{a}_s + 2k(\theta^* - \theta_s^d) = 0$$

$$\Rightarrow \theta_s^d = \frac{-\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^*\right)\right)'\boldsymbol{a}_s + (\mathbf{1}\theta_s)'\boldsymbol{C}\boldsymbol{a}_s + k\theta^*}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}$$

Therefore we get the optimal declaration for an optimizing stubborn.

$$\theta_s^d = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta_s - \frac{\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\mathbf{1}\right)\right)'\boldsymbol{a}_s - k}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta^*$$

Substituting θ_s^d in the steady state opinion vector equation we get exactly (14).

$\square$

A.6 Proof of Proposition 4

To prove that the consensus time of (17) is in the order of $\lambda_2^{\mathcal{A}}$ exponentially we use the following well-know theorem

Theorem 6 (Perron-Frobenius) *Let the eigenvectors be chosen so that $\boldsymbol{\nu}_i' \boldsymbol{v}_i = 1$, where $\boldsymbol{\nu}_i'$ is the left eigenvector and $\boldsymbol{v}_i$ is the right eigenvector. λ_1 and λ_2 the first and second largest eigenvalue and r_2 the algebraic multiplicity associated with λ_2. Then we get*

$$\mathcal{A}^n = (\lambda_1^{\mathcal{A}})^n \boldsymbol{v}_i \boldsymbol{\nu}_i' + o(n^{r_2-1}|\lambda_2^{\mathcal{A}}|^n)$$

Corollary 6.1 *Since $\mathcal{A}$ is a stochastic aperiodic matrix $\lambda_1^{\mathcal{A}} = 1$ and $\boldsymbol{v} = \mathbf{1}$ if the algebraic multiplicity associated with $\lambda_2^{\mathcal{A}}$ r_2 is equal to 1, then*

$$\mathcal{A}^n = \mathbf{1}\boldsymbol{\nu}_i' + o(|\lambda_2^{\mathcal{A}}|^n)$$

a smaller second-largest eigenvalue directly corresponds to a higher rate of convergence.

$\Rightarrow CT(\epsilon, \mathcal{G})$ is in the order of $\lambda_2^{\mathcal{A}}$ exponentially.

Before to prove inequality (19) we have to introduce the definition of Laplacian matrix and the result known as Cheeger's inequality.[34]

Definition (Laplacian Matrix) *A matrix $\mathcal{L} := (l_{ij}) \in \mathbb{R}^{n\times n}$ is a Laplacian Matrix $\mathcal{L}$ iff*

1. $l_{i,j} \leq 0, \quad j \neq i$
2. $\sum_{j=1}^{n} l_{i,j} = 0, \quad i = 1, 2, ..., n$

The Laplacian Matrix can be computed as the difference between the diagonal degree matrix and the adjacency matrix.

Cheeger's Inequality (Chung, 1996) *If $\lambda_{n-2}^{\mathcal{L}^{\mathcal{A}}}$ is the second smallest eigenvalue of the Laplacian of the graph $\mathcal{G}(N, \mathcal{A})$, then:*

$$\frac{\phi(\mathcal{G})^2}{2} \leq \lambda_{n-2}^{\mathcal{L}^{\mathcal{A}}} \leq 2\phi(\mathcal{G})$$

$\mathcal{A}$ is a row-stochastic matrix, thus his Laplacian is $\mathcal{L}^{\mathcal{A}} = \boldsymbol{I} - \mathcal{A}$ and its the second smallest

[34] We refer to Agaev and Chebotarev (2005) for the discussion about non-symmetric Laplacian matrices and to Chung (1996) for Cheeger's inequality.

eigenvalue is nothing but $\lambda_{n-2}^{\mathcal{L}\boldsymbol{\mathcal{A}}} = 1 - \lambda_2^{\boldsymbol{\mathcal{A}}}$, where $\lambda_2^{\boldsymbol{\mathcal{A}}}$ is second largest eigenvalue of the adjacency matrix. Therefore

$$\Rightarrow \frac{\phi(\mathcal{G})^2}{2} \leq 1 - \lambda_2^{\boldsymbol{\mathcal{A}}} \leq 2\phi(\mathcal{G})$$

□

A.7 Proof of Proposition 5

Iterating the process (17) we get

$$\begin{aligned}
\boldsymbol{\mu}_{t+T} &= \boldsymbol{A}^T \boldsymbol{\mu}_t + \sum_{t=0}^{T-1} \boldsymbol{A}^t \boldsymbol{D}\boldsymbol{\theta}^* \\
\boldsymbol{\mu}_{t+T} &= \boldsymbol{A}^T \boldsymbol{\mu}_t + \frac{\boldsymbol{I} - \boldsymbol{A}^T}{\boldsymbol{I} - \boldsymbol{A}} \boldsymbol{D}\boldsymbol{\theta}^* \\
\boldsymbol{\mu}_{t+T} &= \boldsymbol{A}^T \boldsymbol{\mu}_t + (\boldsymbol{I} - \boldsymbol{A}^T)\boldsymbol{\theta}^* \\
\Rightarrow \boldsymbol{\mu}_{t+T} - \boldsymbol{\theta}^* &= \boldsymbol{A}^T(\boldsymbol{\mu}_t - \boldsymbol{\theta}^*)
\end{aligned}$$

Thus depends on how fast is $\boldsymbol{A}^T \to 0$. If $\boldsymbol{A}$ is diagonalizable then, we can define $\boldsymbol{U}$ as the eigenvector matrix and $\boldsymbol{\Lambda}$ as the diagonal matrix with eigenvalues on its main diagonal.

$$\boldsymbol{A} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^{-1}$$

is the eigendecomposition of $\boldsymbol{A}$. Therefore iterating it T times we get

$$\boldsymbol{A}^T = \boldsymbol{U}\boldsymbol{\Lambda}^T\boldsymbol{U}^{-1}$$

$$\begin{aligned}
\|\boldsymbol{\mu}_{t+T} - \boldsymbol{\theta}^*\| &= \|\boldsymbol{A}^T(\boldsymbol{\mu}_t - \boldsymbol{\theta}^*)\| \\
&= \|\boldsymbol{U}\boldsymbol{\Lambda}^T\boldsymbol{U}^{-1}(\boldsymbol{\mu}_t - \boldsymbol{\theta}^*)\| \\
&\leq \sum_{j=1}^{n} |\lambda_j^{\boldsymbol{A}}|^T \|\boldsymbol{U}\| \|\boldsymbol{U}^{-1}\| \|(\boldsymbol{\mu}_t - \boldsymbol{\theta}^*)\| \\
&\leq |\lambda_1^{\boldsymbol{A}}|^T \|\boldsymbol{U}\| \|\boldsymbol{U}^{-1}\|
\end{aligned}$$

Where the last inequality stem from (16), and $\|\boldsymbol{U}\|\|\boldsymbol{U}^{-1}\| = \kappa(U)$ is the condition number of the eigenbasis U. Moreover, if

$$T \geq \frac{log(\epsilon/(\kappa(U)))}{log(|\lambda_1^{\boldsymbol{A}}|)}$$

Then

$$\|\boldsymbol{\mu}_{t+T} - \boldsymbol{\theta}^*\| \leq \epsilon$$

Therefore

$$LT(\epsilon, \mathcal{G}) \leq \lceil \frac{log(\epsilon/(\kappa(U))}{log(|\lambda_1^{\boldsymbol{A}}|)} \rceil$$

The $LT(\epsilon, \mathcal{G})$ depends on eigenvalues and eigenvectors. We can, thus, conclude that $LT(\epsilon, \mathcal{G})$ is of the order of the higher eigenvalue, exponentially. And positively depends on the condition number of the eigenvector basis. Notice that all eigenvalues have a short-run effect, that decays over time according to the their absolute values.

We now prove the inequality (21) in Proposition 5. Let us define as $\bar{d}^{\boldsymbol{A}}$ and $d_{\max}^{\boldsymbol{A}}$ as the average and the maximum degree, respectively

- Lower bound

  Using the Rayleigh quotient (Horn and Johnson, 1985)

$$\frac{\mathbf{1}'\boldsymbol{A}\mathbf{1}}{\mathbf{1}'\mathbf{1}} = \frac{\sum_{ij} a_{ij}}{n} = \frac{\sum_i d_i^{\boldsymbol{A}}}{n} = \bar{d}^{\boldsymbol{A}} \leq \lambda_1^{\boldsymbol{A}} \tag{28}$$

- Upper bound

  Let $\boldsymbol{\nu}_1$ be an eigenvector belonging to $\lambda_1^{\boldsymbol{A}}$ and ν_{1i} be the entry with largest absolute value. Then

$$\lambda_1^{\boldsymbol{A}}|\nu_{1i}| = \sum_j a_{ij}|\nu_{1i}| \leq d_{\max}^{\boldsymbol{A}}|\nu_{1i}|$$

$$\Rightarrow \lambda_1^{\boldsymbol{A}} = \sum_j a_{ij} \leq d_{\max}^{\boldsymbol{A}} \tag{29}$$

Putting together (28) and (29) we finally get

$$\Rightarrow \bar{d}^{\boldsymbol{A}} \leq \lambda_1^{\boldsymbol{A}} \leq d_{\max}^{\boldsymbol{A}}$$

Since $\boldsymbol{A} = \boldsymbol{\mathcal{A}} - \boldsymbol{D}$, then the degree of a generic agent i is $d_i = \sum_i a_{ij} = 1 - a_{ii}$. Thus, the average degree $\bar{d}^{\boldsymbol{A}} = 1 - \sum_i \frac{a_{ii}}{n} = 1 - \bar{a}_{ii}$ and the maximum degree of $\boldsymbol{A}$ is $d_{\max}^{\boldsymbol{A}} = \max_i\{1 - a_{ii}\} = 1 - \min_i\{a_{ii}\}$. Therefore, for a graph described by the adjacency matrix $\boldsymbol{A}$

$$1 - \bar{a}_{ii} \leq \lambda_1^{\boldsymbol{A}} \leq 1 - \min_i\{a_{ii}\}$$

□

A.8 Proof of Corollary 5.1

In general, by Cheeger's inequality we have that

$$2\mathcal{L}^{\boldsymbol{\mathcal{A}}}_{n-2} \leq \phi(\mathcal{G}) \leq 2\sqrt{\lambda^{\mathcal{L}\boldsymbol{\mathcal{A}}}_{n-2}}$$

Namely, $\phi(\mathcal{G})$ is increasing in $\lambda^{\mathcal{L}\boldsymbol{\mathcal{A}}}_{n-2} = 1 - \lambda^{\boldsymbol{\mathcal{A}}}_2$. Therefore we can conclude that

$$\phi(\mathcal{G}_1) \geq \phi(\mathcal{G}_2) \Longrightarrow \lambda_2^{\boldsymbol{\mathcal{A}}_1} \leq \lambda_2^{\boldsymbol{\mathcal{A}}_2} \Longrightarrow CT(\epsilon, \mathcal{G}_1) \leq CT(\epsilon, \mathcal{G}_2) \tag{30}$$

Where the last implication stem from Proposition 4.

If $\boldsymbol{\mathcal{A}}_1 = \alpha \boldsymbol{I} + \boldsymbol{A}_1$ and $\boldsymbol{\mathcal{A}}_2 = \alpha \boldsymbol{I} + \boldsymbol{A}_2$ then, by Proposition 5, we know that $\lambda_1^{\boldsymbol{A}_1} = \lambda_1^{\boldsymbol{A}_2} = 1 - \alpha$. Therefore the first eigenvalues does not tell us which network converge faster to $\boldsymbol{\theta}^*$. We know, from the proof of Proposition 5 that $LT(\epsilon, \mathcal{G})$ depend on eigenvalues of $\boldsymbol{A}$ and not $\boldsymbol{\mathcal{A}}$. Thus, since $\lambda_1^{\boldsymbol{A}_1} = \lambda_1^{\boldsymbol{A}_2} = 1 - \alpha$ we can say that

$$LT(\epsilon, \mathcal{G}_1) \leq LT(\epsilon, \mathcal{G}_2) \iff \lambda_2^{\boldsymbol{A}_1} \leq \lambda_2^{\boldsymbol{A}_2} \tag{31}$$

Lemma 2 (Horn and Johnson, 1985) *Given two commuting matrix $\boldsymbol{C}$ and $\boldsymbol{D}$, there exists a unitary matrix $\boldsymbol{U}$ such that $\boldsymbol{U}^{-1}\boldsymbol{C}\boldsymbol{U} = \Lambda^{\boldsymbol{C}}$ and $\boldsymbol{U}^{-1}\boldsymbol{D}\boldsymbol{U} = \Lambda^{\boldsymbol{D}}$. where $\Lambda^{\boldsymbol{C}}, \Lambda^{\boldsymbol{D}}$ are diagonal matrices with eigenvalues as elements. Thus we get*

$$\boldsymbol{C} + \boldsymbol{D} = \boldsymbol{U}(\Lambda^{\boldsymbol{C}} + \lambda^{\boldsymbol{D}})\boldsymbol{U}^{-1}$$

Applying this lemma to both $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{A}}_2$ and considering only the second largest eigenvalue, we obtain that

$$\lambda_2^{\boldsymbol{\mathcal{A}}_1} = \alpha + \lambda_2^{\boldsymbol{A}_1}, \; \lambda_2^{\boldsymbol{\mathcal{A}}_2} = \alpha + \lambda_2^{\boldsymbol{A}_2}$$

Therefore if

$$\lambda_2^{\boldsymbol{\mathcal{A}}_1} \leq \lambda_2^{\boldsymbol{\mathcal{A}}_2} \iff \lambda_2^{\boldsymbol{A}_1} \leq \lambda_2^{\boldsymbol{A}_2} \tag{32}$$

Using (30), (31) and (32) togheter we get exactly the result of Corollary 5.1.

□

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B Supplementary Materials

B.1 Convergence toward the Truth

Let us consider a network with a malevolent stubborn agent, the "spreader", s who affects the opinion dynamics declaring $\theta_s \neq \theta^*$ and a benevolent policymaker p (still stubborn) who want citizens to be as more informed as possible and thus to minimize the distance of steady state opinion vectors to the truth[35]

$$u_p(\boldsymbol{\mu}) := -(\boldsymbol{\mu} - \mathbf{1}\theta^*)^2 \tag{33}$$

Since it is not always possible to directly affect the network structure, we wonder if there is an action (declaration of θ_p) that the policymaker p can do to promote the spread of truth against the presence of stubborn agents in the society. It is clear that (30) is maximized when all agents have opinions equal to θ^*. In the next proposition, we show that if the influence of the two stubborn (p=policymaker and s=spreader) is not symmetric, then it is not possible to reach exactly the truth

Proposition 7 *If the two stubborn agents have different influence over agents, $\boldsymbol{a}_p \neq \boldsymbol{a}_s \neq 0$, and $\theta_s \neq \theta^*$ then*

$$\nexists\, \theta_g \ : \ \boldsymbol{\mu} = \boldsymbol{\theta^*}$$

If stubborn agents have equal influence over agents, $\boldsymbol{a}_p = \boldsymbol{a}_s$, and one among them declare

$$\theta_p = 2\theta^* - \theta_s$$

then all agents in the society learn the truth:

$$\boldsymbol{\mu} = \boldsymbol{\theta^*}$$

Notice that this results can be extended to a society with more than two stubborn, in the appendix we provide the characterization of the result with many stubborn.

Even if, whenever $\boldsymbol{a}_p \neq \boldsymbol{a}_s$ it is not possible for a policymaker to chose θ_p such that $\boldsymbol{\mu} = \boldsymbol{\theta^*}$ it is still possible to get close enough to the truth by declaring an opinion that minimizes the distance with the steady state opinions' vector.

Proposition 8 *If all agents have IR of past signals and there a malevolent spreader s with fixed opinion $\theta_s \neq \theta^*$ and a benevolent policymaker p with utility function (33), then*

[35] Notice that at this stage we assume the stubborn agent to be naive, namely the opinion $\theta_s \neq \theta^*$ is not the result of a maximization process, but it is exogenous.

p would declare the following opinion

$$\theta_p = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_p - \left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1}\right)\right)'\boldsymbol{a}_p}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\theta^* - \frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\theta_s$$

then steady state vector of beliefs (opinions) is

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1} + \boldsymbol{a}_p\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_p - \left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1}\right)\right)'\boldsymbol{a}_p}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\right)\theta^* + \left(\boldsymbol{a}_s - \boldsymbol{a}_p\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\right)\theta_s\right) \tag{34}$$

If the stubborn is sophisticated as in Section 3.3, the policy maker has to keep that into account. To have more tractable results we assume, without loss of generality, that $\theta_p = \theta^* = 0$

Proposition 9 *If all agents have "imperfect recall" and there is a sophisticated malevolent spreader s that solves the problem in (12) and a benevolent policymaker p with utility function (33) where $\theta_p = \theta^* = 0$, then the declared opinions are*

$$\begin{cases} \theta_s^d = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p+k)}{j}\theta_s \\ \theta_p^d = -\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p+k}\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p+k)}{j}\theta_s \end{cases} \tag{35}$$

where $j = k(k + \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + \boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p) + \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p - \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s$ is a constant.

Then steady state vector of beliefs (opinions) is

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k)}{j}\left(\boldsymbol{a}_s - \boldsymbol{a}_p\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k}\right)\right)\theta_s \tag{36}$$

We now provide proofs of proposition in this section.

B.2 Proofs of Supplementary Materials

B.2.1 Proof of Proposition 7

Notice that since $\boldsymbol{\mathcal{A}}$ is stochastic matrix then $(\boldsymbol{I}-\boldsymbol{A})\mathbf{1} = \boldsymbol{D}\mathbf{1} + \boldsymbol{a}_{s1} + \boldsymbol{a}_{s2}$, then

- If $\boldsymbol{a}_{s1} = \boldsymbol{a}_{s2} = 0$

$$\boldsymbol{\mu} = (\boldsymbol{I} - \boldsymbol{D}\boldsymbol{G} - \boldsymbol{A})^{-1}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\boldsymbol{\theta}^*\right)$$

where $(\boldsymbol{I}-\boldsymbol{DG}-\boldsymbol{A})\mathbf{1} = (\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1})$, thus

$$\boldsymbol{\mu} = (\boldsymbol{I}-\boldsymbol{DG}-\boldsymbol{A})^{-1}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1}\right)\theta^{*}$$

$$\Rightarrow \boldsymbol{\mu} = \underbrace{(\boldsymbol{D}-\boldsymbol{DG})^{-1}\left(\boldsymbol{D}-\boldsymbol{DG}\right)}_{\boldsymbol{I}}\mathbf{1}\theta^{*} = \boldsymbol{\theta}^{*}$$

- If $\boldsymbol{\theta}_{s1} = \boldsymbol{\theta}_{s2} = \theta^{*}$

$$\boldsymbol{\mu} = (\boldsymbol{I}-\boldsymbol{DG}-\boldsymbol{A})^{-1}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\boldsymbol{\theta}^{*} + \boldsymbol{a}_{s1}\theta^{*} + \boldsymbol{a}_{s2}\theta^{*}\right)$$

where $(\boldsymbol{I}-\boldsymbol{DG}-\boldsymbol{A})\mathbf{1} = (\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1} + \boldsymbol{a}_{s1} + \boldsymbol{a}_{s2})$, thus

$$\boldsymbol{\mu} = (\boldsymbol{D} + \boldsymbol{a}_{s1} + \boldsymbol{a}_{s2} - \boldsymbol{DG})^{-1}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1} + \boldsymbol{a}_{s1} + \boldsymbol{a}_{s2}\right)\theta^{*}$$

$$\Rightarrow \boldsymbol{\mu} = \underbrace{(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G}) + \boldsymbol{a}_{s1} + \boldsymbol{a}_{s2})^{-1}((\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G}) + \boldsymbol{a}_{s1} + \boldsymbol{a}_{s2})}_{\boldsymbol{I}}\boldsymbol{\theta}^{*} = \theta^{*}$$

- $\boldsymbol{\mu} = \boldsymbol{\theta^{*}}$ if

$$\boldsymbol{a}_{s1}\theta_{s1} + \boldsymbol{a}_{s2}\theta_{s2} = (\boldsymbol{a}_{s1} + \boldsymbol{a}_{s2})\,\theta^{*}$$

$$\boldsymbol{a}_{s1}\theta_{s1} = (\boldsymbol{a}_{s1} + \boldsymbol{a}_{s2})\,\theta^{*} - \boldsymbol{a}_{s2}\theta_{s2}$$

$$\boldsymbol{a}_{s1}\theta_{s1} = \boldsymbol{a}_{s1}\theta^{*} + \boldsymbol{a}_{s2}(\theta^{*} - \theta_{s2})$$

If θ_{s1} it is a scalar, the system has only one solution

$$\theta_{s1} = 2\theta^{*} - \theta_{s2}$$

if and only if $\boldsymbol{a}_{s1} = \boldsymbol{a}_{s2}$.

On the other hand, if $\boldsymbol{a}_{s1} \neq \boldsymbol{a}_{s2}$ if θ_{s1} is a scalar is not possible for one of the two stubborn to compensate the distortion created by the other.

(Double check) If $\boldsymbol{a}_{s1} = \boldsymbol{a}_{s2} = \boldsymbol{a}_{s}$ and $\theta_{s1} = 2\theta^{*} - \theta_{s2}$

$$\boldsymbol{\mu} = (\boldsymbol{I}-\boldsymbol{DG}-\boldsymbol{A})^{-1}(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\boldsymbol{\theta^{*}} + \boldsymbol{a}_{s1}\left(2\theta^{*} - \theta_{s2}\right) + \boldsymbol{a}_{s2}\theta_{s_2})$$

$$\boldsymbol{\mu} = (\boldsymbol{I}-\boldsymbol{DG}-\boldsymbol{A})^{-1}(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\boldsymbol{\theta^{*}} + 2\boldsymbol{a}_{s}\theta^{*})$$

$$\boldsymbol{\mu} = (\boldsymbol{I}-\boldsymbol{DG}-\boldsymbol{A})^{-1}(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1} + 2\boldsymbol{a}_{s})\theta^{*}$$

which is equivalent to

$$\boldsymbol{\mu} = (\boldsymbol{D} + 2\boldsymbol{a}_s - \boldsymbol{DG})^{-1}(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\mathbf{1} + 2\boldsymbol{a}_s)\theta^*$$

The first two bullet points discuss the conditions under which the truth is always reached while the third bullet point prove Proposition 7. Now we generalize the result to more than 2 stubborn.

If the cardinality of the set of stubborn agents is S and $\boldsymbol{a}_{s1} = \boldsymbol{a}_{s2} = ... = \boldsymbol{a}_{ss} = \boldsymbol{a}_s$ then $\boldsymbol{\mu} = \boldsymbol{\theta^*}$ if

$$\boldsymbol{a}_s\theta_{s1} + \sum_{s=2}^{S} \boldsymbol{a}_s\theta_{ss} = \left(\sum_{s=1}^{S} \boldsymbol{a}_{ss}\right)\theta^*$$

$$\Rightarrow \boldsymbol{a}_s\theta_{s1} = \left(\boldsymbol{a}_s + \sum_{s=2}^{S} \boldsymbol{a}_{ss}\right)\theta^* - \sum_{s=2}^{S} \boldsymbol{a}_{ss}\theta_{ss}$$

$$\Rightarrow \boldsymbol{a}_s\theta_{s1} = \boldsymbol{a}_s\theta^* + \sum_{s=2}^{S} \boldsymbol{a}_{ss}(\theta^* - \theta_{ss})$$

$$\Rightarrow \theta_s = \theta^* + \sum_{s=2}^{S} \frac{\boldsymbol{a}_{ss}(\theta^* - \theta_{ss})}{\boldsymbol{a}_{s1}}$$

□

B.2.2 Proof Proposition 8

Maximizing the utility function (33)

$$\max_{\theta_p} -(\boldsymbol{\mu} - \mathbf{1}\theta^*)^2$$

$$\max_{\theta_p} -\boldsymbol{\mu}'\boldsymbol{\mu} + 2(\mathbf{1}\theta^*)'\boldsymbol{\mu}$$

since

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^* + \boldsymbol{a}_p\theta_p + \boldsymbol{a}_s\theta_s\right)$$

substituting $\boldsymbol{\mu}$ in the problem and considering only elements depending on θ_p we get

$$\max_{\theta_p} \; -2\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} + \boldsymbol{G})\boldsymbol{\theta}^*\right)\right)' \boldsymbol{a}_p\theta_p - 2\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\theta_p\theta_s - \boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p\theta_p^2 + 2(\mathbf{1}\theta^*)'\boldsymbol{C}\boldsymbol{a}_p\theta_p$$

we now solve the First Order Condition

$$-2\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\boldsymbol{\theta}^*\right)\right)' \boldsymbol{a}_p - 2\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\theta_s - 2\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p\theta_p + 2(\mathbf{1}\theta^*)'\boldsymbol{C}\boldsymbol{a}_p = 0$$

$$-\left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{G})\mathbf{1}\right)\right)' \boldsymbol{a}_p\theta^* - \boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\theta_s - \boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p\theta_p + \mathbf{1}'\boldsymbol{C}\boldsymbol{a}_p\theta^* = 0$$

Thus the optimal policy maker's declaration is

$$\theta_p = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_p - \left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1}\right)\right)'\boldsymbol{a}_p}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\theta^* - \frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\theta_s$$

Substituting in $\boldsymbol{\mu}$ we get exactly

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\boldsymbol{\theta}^* + \boldsymbol{a}_p\left(\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_p - \left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I}+\boldsymbol{G})\mathbf{1}\right)\right)'\boldsymbol{a}_p}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\theta^* - \frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\theta_s\right) + \boldsymbol{a}_s\theta_s\right)$$

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1} + \boldsymbol{a}_p\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_p - \left(\boldsymbol{C}\left(\boldsymbol{D}(\boldsymbol{I}-\boldsymbol{G})\mathbf{1}\right)\right)'\boldsymbol{a}_p}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\right)\theta^* + \left(\boldsymbol{a}_s - \boldsymbol{a}_p\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}\right)\theta_s\right) \tag{37}$$

□

B.2.3 Proof Proposition 9

There are two stubborn, one controlled by the policy maker p and spreader of misinformation s.

$$\max_{\theta_p^d}\ u_s(\boldsymbol{\mu}):\ -(\boldsymbol{\mu}-\mathbf{1}\theta_p)^2 - k(\theta^* - \theta_p^d)^2$$

We assume, without loss of generality, that $\theta_p = \theta^* = 0$

$$\max_{\theta_p^d}\ u_s(\boldsymbol{\mu}):\ -(\boldsymbol{\mu}-\mathbf{0})^2 - k(-\theta_p^d)^2$$

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\boldsymbol{a}_p\theta_p^d + \boldsymbol{a}_s\theta_s^d\right) \tag{38}$$

$$\max_{\theta_p^d}\ -\boldsymbol{\mu}'\boldsymbol{\mu} - k(-\theta_p^d)^2$$

$$\max_{\theta_p^d}\ -2\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\theta_p^d\theta_s^d - \boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p\theta_p^{d2} - k(-\theta_p^d)^2$$

F.O.C.

$$-\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\theta_s^d - \boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p\theta_p^d - k\theta_p^d = 0$$

Solving for θ_p^d

$$\theta_p^d = -\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k}\theta_s^d$$

by symmetry

$$\theta_s^d = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta_s - \frac{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta_p^d$$

Substituting θ_p^d

$$\Rightarrow \theta_s^d = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta_s + \frac{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k}\theta_s^d$$

Solving for θ_s^d

$$\theta_s^d \frac{k(k + \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + \boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p) + \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p - \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{(\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k)(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k)} = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + k}\theta_s$$

Therefore

$$\Rightarrow \theta_s^d = \frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k)}{j}\theta_s \tag{39}$$

where $j = k(k + \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s + \boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p) + \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p - \boldsymbol{a}_s'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s$ is a constant.

$$\theta_p^d = -\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k}\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k)}{j}\theta_s \tag{40}$$

θ_s^d and θ_p^d are both decreasing in k.

Substituting (39) and (40) into (38) we obtain

$$\boldsymbol{\mu} = \boldsymbol{C}\left(-\boldsymbol{a}_p\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k}\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k)}{j} + \boldsymbol{a}_s\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k)}{j}\right)\theta_s$$

$$\boldsymbol{\mu} = \boldsymbol{C}\left(\frac{\mathbf{1}'\boldsymbol{C}\boldsymbol{a}_s(\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k)}{j}\left(\boldsymbol{a}_s - \boldsymbol{a}_p\frac{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_s}{\boldsymbol{a}_p'\boldsymbol{C}'\boldsymbol{C}\boldsymbol{a}_p + k}\right)\right)\theta_s$$