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Abstract
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Keywords
Bayesian Confirmation Measures, symmetries, dihedral group, desirability

JEL Codes
C10, C44, C80

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Abstract

Many Bayesian Confirmation Measures have been proposed so far. They are used to assess the degree to which an evidence (or premise) E supports or contradicts an hypothesis (or conclusion) H, making use of prior probability $P(H)$, posterior probability $P(H|E)$ and of probability of evidence $P(E)$. Many kinds of comparisons of those measures have already been made. Here we focus on symmetry properties of confirmation measures, which are partly inspired by classical geometric symmetries. We define symmetries relating them to the dihedral group of symmetries of the square, determining the symmetries that can coexist and reconsidering desirable/undesirable symmetry properties for a Bayesian Confirmation Measure.

KEYWORDS: Bayesian Confirmation Measures, symmetries, dihedral group, desirability.
J.E.L. CLASSIFICATION: C10, C44, C80

1 Introduction

Inductive rules, $E \to H$, are a way to express relationships in a dataset, meaning that the knowledge of E supports conclusion H, and can be supported by data with different intensities.

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Bayesian Confirmation Measures (BCMs) are interestingness measures aimed at evaluating the degree to which an evidence E supports or contradicts the conclusion H, using prior probability $P(H)$, posterior probability $P(H|E)$ and $P(E)$, the probability of evidence E. Evidence E may in fact confirm conclusion H when $P(H|E) > P(H)$, or disconfirmed it when $P(H|E) < P(H)$.

It is therefore quite natural to define a measure $c(H, E)$ that satisfies the following Definition (see, e.g., [8], [12]):

Definition 1 *A function c of conclusion H and evidence E, is a Bayesian Confirmation Measure (BCM) when*

$c(H, E) > 0$ *if* $P(H|E) > P(H)$ *(confirmation case)*

$c(H, E) = 0$ *if* $P(H|E) = P(H)$ *(neutrality case)*

$c(H, E) < 0$ *if* $P(H|E) < P(H)$ *(disconfirmation case)*

The study of the analytical properties of BCMs can allow to grasp the differences among measures (see [11] and [15]) but also geometric and visual approaches allow to uncover their properties and to suggest how to select the right one to be used for a specific purpose (see [3], [20] and [21]). Symmetries are, not surprisingly, among the properties that can be better captured by a geometric approach. The first symmetry properties that were defined by Carnap [2], concern the confirmation values attained by the measure when negation of evidence, $\neg E$, is substituted to evidence E, negation of conclusion, $\neg H$, is substituted to conclusion H, or when the roles of evidence and conclusion are inverted. Crupi et al. (see [7]) completed the set of symmetries proposed by Carnap considering all their possible combinations.

Here we propose a group theoretical way to clarify the relationships between the different symmetries for BCMs proposed in the literature. To do that, we define a set Σ of symmetry functions acting on the couples (H, E) of hypothesis and evidence sentences, and then we endow the set with the composition of symmetries obtaining a group structure that we prove to be isomorphic to the dihedral group D_8 (Section 2). Relating the symmetry functions to the symmetry properties of BCMs (Section 3) we uncover the possible subsets of concurrent symmetries a BCM can display (Section 4), providing examples of BCMs that possess exactly those subsets of symmetries. The group structure of the set Σ provides both quick paths to the study of all the symmetry properties of a BCM and ways to analyse the desirability/undesirability (see [7], [9], [14], [15], [17]) of the symmetries they display (Section 5).

2 The group of symmetry functions defined on hypotesis-evidence couples

Crupi et al. [7] define symmetry functions as applied to a couple hypothesis-evidence (H, E), i.e. an ordered couple of sentences H (Hypothesis) and E (Evidence) where H and E belong to the set of sentences Γ which is supposed to be closed under negation and conjunction.
Following [7] we consider three basic symmetry functions $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{I}$ defined in $\Gamma \times \Gamma$:

$\mathcal{E}$ negation of Evidence: $\mathcal{E}(H, E) = (H, \neg E)$;

$\mathcal{H}$ negation of Hypothesis: $\mathcal{H}(H, E) = (\neg H, E)$;

$\mathcal{I}$ inversion of Evidence and Hypothesis: $\mathcal{I}(H, E) = (E, H)$.

In other words, the rule $E \to H$ is changed into $\neg E \to H$ by $\mathcal{E}$, into $E \to \neg H$ by $\mathcal{H}$ and into $H \to E$ by $\mathcal{I}$.
Symmetry functions $\mathcal{E}$, $\mathcal{H}$ and $\mathcal{I}$ can be easily composed so as to obtain the other four symmetry functions proposed in [7]:

$\mathcal{EH}$ negation of Evidence and Hypothesis:

$$\mathcal{EH}(H, E) = \mathcal{H}(\mathcal{E}(H, E)) = (\neg H, \neg E);$$

$\mathcal{EI}$ inversion follows negation of Evidence:

$$\mathcal{EI}(H, E) = \mathcal{I}(\mathcal{E}(H, E)) = (\neg E, H);$$

$\mathcal{HI}$ inversion follows negation of Hypothesis:

$$\mathcal{HI}(H, E) = \mathcal{I}(\mathcal{H}(H, E)) = (E, \neg H);$$

$\mathcal{EHI}$ inversion follows negation of Evidence and Hypothesis:

$$\mathcal{EHI}(H, E) = \mathcal{I}(\mathcal{H}(\mathcal{E}(H, E))) = (\neg E, \neg H).$$

Let us add to the seven above recalled symmetry functions the identity function σ_0 (for which $\sigma_0(H, E) = (H, E)$ for all (H, E)). We obtain a set of symmetry functions

$$\Sigma = \{\sigma_0, \mathcal{E}, \mathcal{H}, \mathcal{I}, \mathcal{EH}, \mathcal{EI}, \mathcal{HI}, \mathcal{EHI}\}$$

that can be endowed with the composition of symmetries operation $\otimes$ as detailed by the Cayley table (Table 1). Observe that in position (i, j) of the Cayley table we put the symmetry function $\sigma_i \otimes \sigma_j$ obtained as $\sigma_i \otimes \sigma_j(H, E) = \sigma_j(\sigma_i(H, E))$.
Table 1 reveals that $(\Sigma, \otimes)$ is a non-commutative group (for non-commutativity see, e.g., that $\mathcal{E} \otimes \mathcal{I}$ does not coincide with $\mathcal{I} \otimes \mathcal{E}$) that will be proved to be isomorphic

Table 1: Cayley table of $(\Sigma, \otimes)$; symmetry in the first column operates first.

$\otimes$	$\mathcal{E}$	$\mathcal{H}$	$\mathcal{I}$	$\mathcal{EH}$	$\mathcal{EI}$	$\mathcal{HI}$	$\mathcal{EHI}$
$\mathcal{E}$	σ_0	$\mathcal{EH}$	$\mathcal{EI}$	$\mathcal{H}$	$\mathcal{I}$	$\mathcal{EHI}$	$\mathcal{HI}$
$\mathcal{H}$	$\mathcal{EH}$	σ_0	$\mathcal{HI}$	$\mathcal{E}$	$\mathcal{EHI}$	$\mathcal{I}$	$\mathcal{EI}$
$\mathcal{I}$	$\mathcal{HI}$	$\mathcal{EI}$	σ_0	$\mathcal{EHI}$	$\mathcal{H}$	$\mathcal{E}$	$\mathcal{EH}$
$\mathcal{EH}$	$\mathcal{H}$	$\mathcal{E}$	$\mathcal{EHI}$	σ_0	$\mathcal{HI}$	$\mathcal{EI}$	$\mathcal{I}$
$\mathcal{EI}$	$\mathcal{EHI}$	$\mathcal{I}$	$\mathcal{E}$	$\mathcal{HI}$	$\mathcal{EH}$	σ_0	$\mathcal{H}$
$\mathcal{HI}$	$\mathcal{I}$	$\mathcal{EHI}$	$\mathcal{H}$	$\mathcal{EI}$	σ_0	$\mathcal{EH}$	$\mathcal{E}$
$\mathcal{EHI}$	$\mathcal{EI}$	$\mathcal{HI}$	$\mathcal{EH}$	$\mathcal{I}$	$\mathcal{E}$	$\mathcal{H}$	σ_0

Table 2: Isomomorphism of D_8 into Σ.

Dihedral group D_8	$\rightarrow$	***Symmetry functions group*** $(\Sigma, \otimes)$
e	$\rightarrow$	σ_0
a	$\rightarrow$	$\mathcal{HI}$
a^2	$\rightarrow$	$\mathcal{EH}$
a^3	$\rightarrow$	$\mathcal{EI}$
x	$\rightarrow$	$\mathcal{EHI}$
ax	$\rightarrow$	$\mathcal{H}$
a^2x	$\rightarrow$	$\mathcal{I}$
a^3x	$\rightarrow$	$\mathcal{E}$

to the dihedral group D_8. [1] To relate $(\Sigma, \otimes)$ with the dihedral group D_8 we consider the map defined in Table 2, where e is the identity in D_8, while the usual geometric interpretation of a is the counter-clockwise rotation by $\pi/2$ and of x is the reflection about a fixed line.

The map defined in Table 2 is an isomorphism, as can be easily proved by comparing the Cayley tables of the two Groups: in Table 3 we report the Cayley's table for D_8.

Therefore, adapting the classical lattice representation of the structure of the dihedral group D_8, we can also depict the subgroup structure of the Symmetry functions subgroup $(\Sigma, \otimes)$ as in Figure 1. The structure of the subgroup lattice will be used in Section 5 to suggest an apt way to explore symmetry properties of a BCM, in particular considering the Frattini subgroup.[2]

[1]The dihedral group can be formally defined as generated by two elements a and x such that $a^4 = x^2 = e$ and $xa = a^{-1}x$, see, e.g., [1].

[2]The Frattini subgroup is the intersection of all maximal subgroups.

Table 3: Cayley table of D_8; symmetry in the first column operates first

	a^3x	ax	a^2x	a^2	a^3	a	x
a^3x	e	a^2	a^3	ax	a^2x	x	a
ax	a^2	e	a	a^3x	x	a^2x	a^3
a^2x	a	a^3	e	x	ax	a^3x	a^2
a^2	ax	a^3x	x	e	a	a^3	a^2x
a^3	x	a^2x	a^3x	a	a^2	e	ax
a	a^2x	x	ax	a^3	e	a^2	a^3x
x	a^3	a	a^2	a^2x	a^3x	ax	e

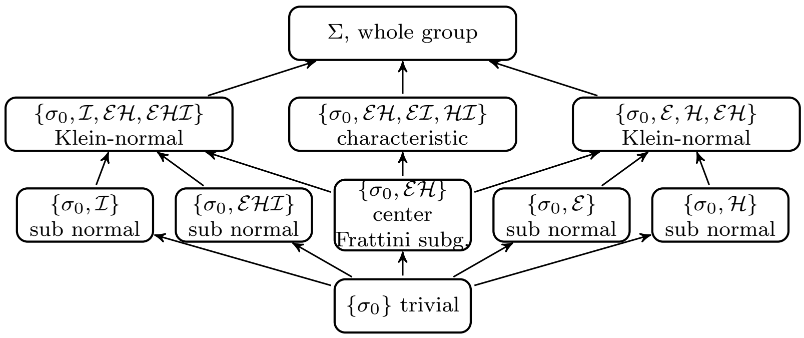

Figure 1: Lattice of subgroups of group $(\Sigma, \otimes)$

3 Symmetries of Bayesian Confirmation Measures

The symmetry properties of BCMs have already been related to geometric symmetries, see the recent papers [3], [4] and [20]. In particular, the geometric interpretation proposed in [3] is set in the so-called Confirmation Space which is defined by the two dimensions $x = P(H|E)$ and $y = P(H)$, where confirmation measures that are IFPD measures [3] can be visualised in a particularly vivid way.

A group theoretic approach has been suggested in [21] treating symmetries as permutations of the elements of 2x2 contingency tables, and compositions of symmetries as compositions of permutations. Changing the point of view, we prefer to consider symmetries as defined by means of logical variations of the involved elements in the inductive rule $E \to H$, that is, on different combinations of H, E and of their negations $\neg H$ and $\neg E$; it becomes therefore interesting to analyse symmetry properties by referring to symmetry functions $\sigma \in \Sigma$. We will see that this way the group structure of $(\Sigma, \otimes)$ can be used to detect in a straightforward manner which symmetry properties of a BCM can coexist. This is another side of the concept of inconsistency in [21]),

[3] In the particular case in which a confirmation measure can be expressed as a function of $P(H|E)$ and $P(H)$ only, it is said to satisfy the Initial and Final Probability Dependence (IFPD) condition [8].

which, in turn, appears to be quite interesting in connection with the debate on the desirability of symmetry properties of a Bayesian Confirmation Measure [7], [17].

Since we want to exploit the group structure of $(\Sigma, \otimes)$ in order to have an in-depth look on symmetry properties of BCMs, we observe first, that each symmetry function $\sigma \in (\Sigma, \otimes)$ can be used to define a corresponding symmetry property S defined on BCMs. For example, a BCM $c(H, E)$ is said to satisfy evidence symmetry if $c(H, E) = -c(H, \neg E)$ (see [7]): alternatively, we can consider the negation of evidence function $\mathcal{E}$ and say that $c(H, E)$ satisfies evidence symmetry if $c(H, E) = -c(\mathcal{E}(H, E))$;likewise, using the negation of hypothesis symmetry function $\mathcal{H}$, it is possible to set the definition of hypothesis symmetry property (see [7]) of a BCM: $c(H, E) = -c(\mathcal{H}(H, E)) = -c(\neg H, E)$.

In general, given a BCM $c(H, E)$ and a symmetry function $\sigma(H, E) \in \Sigma$ we can define a symmetry property of c

$$c(H, E) = sign(\sigma) \cdot c(\sigma(H, E))$$

where $sign(\sigma)$ must be set, given the chosen symmetry function, in order to fulfil the basic sign requirements of a BCM as reported in Definition 1. Nicely, it turns out that the correct sign to be considered is always $(-1)^k$ where k is the number of negations $(\neg)$ contained in the symmetry function definition.

The set of symmetry properties of a BCM (see [7]) can therefore be viewed as originated by the set of symmetry functions Σ, as outlined in Definition 2.

Definition 2 *A confirmation measure c satisfies*

Evidence Symmetry *ES if*

$$c(H, E) = -c(\mathcal{E}(H, E)) = -c(H, \neg E);$$

Hypothesis Symmetry *HS if*

$$c(H, E) = -c(\mathcal{H}(H, E)) = -c(\neg H, E);$$

Evidence Hypothesis Symmetry *EHS if*

$$c(H, E) = c(\mathcal{E}\mathcal{H}(H, E)) = c(\neg H, \neg E);$$

Inversion Symmetry *IS if*

$$c(H, E) = c(\mathcal{I}(H, E)) = c(E, H);$$

Evidence Inversion Symmetry *EIS if*

$$c(H, E) = -c(\mathcal{E}\mathcal{I}(H, E)) = -c(\neg E, H);$$

Hypothesis Inversion Symmetry *HIS if*

$$c(H,E) = -c(\mathcal{HI}(H,E)) = -c(E,\neg H);$$

Evidence Hypothesis Inversion Symmetry *EHIS if*

$$c(H,E) = c(\mathcal{EHI}(H,E)) = c(\neg E,\neg H).$$

We define $\mathbf{\Sigma} = \{ES, HS, EHS, IS, EIS, HIS, EHIS\}$ as the set of all possible symmetries of a BCM.

4 Exploiting the structure of the symmetry functions group

As mentioned, the group structure of $(\Sigma, \otimes)$ can be used to detect which symmetry properties of a BCM can coexist, that is, all the feasible *combinations* of symmetries taken from set $\mathbf{\Sigma}$.

Let us start by considering the way in which new symmetry properties can be detected as soon as we know a couple of them.

Lemma 1 *Given a confirmation measure c satisfying two symmetry properties in* $\mathbf{\Sigma}$*, say* S_1*,* S_2*, with corresponding symmetry functions* σ_1*,* $\sigma_2 \in \Sigma$ *and signs* $(-1)^{k_1}$*,* $(-1)^{k_2}$*, then necessarily c satisfies other two symmetry properties in* $\mathbf{\Sigma}$*, say* S_3*,* S_4*, with sign* $(-1)^{k_1+k_2}$ *and corresponding symmetry functions* $\sigma_3 = \sigma_1 \otimes \sigma_2$ *and* $\sigma_4 = \sigma_2 \otimes \sigma_1$*.*

Proof
Given that c is supposed to satisfy symmetry properties S_1 and S_2, it is $c(H,E) = (-1)^{k_1} c(\sigma_1(H,E))$ and $c(H,E) = (-1)^{k_2} c(\sigma_2(H,E))$ where σ_1 and σ_2 are the corresponding symmetry functions, and k_1, k_2 denote the number of negations $\neg$ contained in the symmetry function definitions.
Then, $c(H,E) = (-1)^{k_1} c(\sigma_1(H,E)) = (-1)^{k_1}(-1)^{k_2} c(\sigma_2(\sigma_1(H,E))$, by repeated application of symmetry hypotheses.
In the same way, $c(H,E) = (-1)^{k_2} c(\sigma_2(H,E)) = (-1)^{k_2}(-1)^{k_1} c(\sigma_1(\sigma_2(H,E))$, so that $c(H,E) = (-1)^{k_1+k_2} c(\sigma_1(\sigma_2(H,E)) = (-1)^{k_1+k_2} c(\sigma_2(\sigma_1(H,E)))$.

■

By the way, observe that from Lemma 1, it follows that if c satisfies two symmetry properties S_1, $S_2 \in \mathbf{\Sigma}$, then with reference to symmetry functions σ_1, $\sigma_2 \in \Sigma$, it is $c(\sigma_1(\sigma_2(H,E))) = c(\sigma_2(\sigma_1(H,E)))$ for all $H, E \in \Gamma$.
A first consequence of the group structure of $(\Sigma, \otimes)$ can now be given in the next Proposition.

Proposition 1 *A Bayesian Confirmation Measure satisfies either 1, 3 or 7 symmetry properties, or no one.*

Proof
From Proposition 1 it follows that if S_1 and S_2 hold, with corresponding symmetry functions σ_1, $\sigma_2 \in \Sigma$, then necessarily also S_3, with corresponding symmetry $\sigma_3 = \sigma_1 \otimes \sigma_2$, and S_4, with corresponding symmetry $\sigma_4 = \sigma_2 \otimes \sigma_1$, must hold, where σ_1, σ_2, σ_3 and σ_4 obviously belong to the same subgroup of $(\Sigma, \otimes)$. By Lagrange's Theorem (see, e.g., [1]) we know that in finite groups the order of a subgroup divides the order of the group and $(\Sigma, \otimes)$ has order 8. Subgroups can therefore have either 1, 2, 4 or 8 elements, identity included. The result follows excluding from each possible subgroup the identity symmetry function since, of course, it does not correspond to any BCM symmetry.

■

The link between symmetry functions and the symmetries of BCMs allows to precisely deduce the symmetry properties that can be concurrently satisfied by a BCM, a result that follows by considering all the possible subgroups of the group of symmetry functions $(\Sigma, \otimes)$ and the corresponding symmetry properties.

Proposition 2 *A Bayesian Confirmation Measures can satisfy exactly one of the following sets of concurrent symmetries:*

(i) *no symmetry property;*

(ii) *exactly one symmetry among* IS, $EHIS$, EHS, ES *and* HS;

(iii) *exactly* $IS \wedge EHS \wedge EHIS$ *or* $EHS \wedge EIS \wedge HIS$ *or* $ES \wedge HS \wedge EHS$;

(iv) *all the symmetry properties* ES, HS, IS, EHS, EIS, HIS, $EHIS$.

Proof
Proposition 1 establishes the cardinality of each subset of concurrent symmetry properties; given the isomorphism of Σ into D_8 it is possible to explicitly write all the possible subgroups of symmetry functions of $(\Sigma, \otimes)$ and therefore the possible corresponding symmetry properties that can be jointly satisfied. The possible subgroups are (see Figure 1):

order 1: $\{\sigma_0\}$,
order 2: $\{\sigma_0, \mathcal{I}\}$, $\{\sigma_0, \mathcal{EHI}\}$, $\{\sigma_0, \mathcal{EH}\}$, $\{\sigma_0, \mathcal{E}\}$, $\{\sigma_0, \mathcal{H}\}$,
order 4: $\{\sigma_0, \mathcal{I}, \mathcal{EH}, \mathcal{EHI}\}$, $\{\sigma_0, \mathcal{EH}, \mathcal{EI}, \mathcal{HI}\}$, $\{\sigma_0, \mathcal{E}, \mathcal{H}, \mathcal{EH}\}$,
order 8: the whole group $\{\sigma_0, \mathcal{E}, \mathcal{H}, \mathcal{I}, \mathcal{EH}, \mathcal{EI}, \mathcal{HI}, \mathcal{EHI}\}$.

It turns out that the symmetry properties that a BCM can simultaneously satisfy are those stated in the Proposition.

■

Given the isomorphism of $(\Sigma, \otimes)$ with D_8, Proposition 2 identifies thus a partition of the set of BCMs into 10 equivalence classes, each partition being defined by the set of satisfied symmetries.

Moreover, the lattice structure of subgroups, reported in Figure 1, suggests quick ways to check the fulfillment of concurrent symmetry properties by a single BCM. For example, considering that $\{\sigma_0, \mathcal{EH}\}$ is the Frattini subgroup of $(\Sigma, \otimes)$, i.e. the intersection of all its maximal subgroups, suggests to explore symmetry properties of a BCM starting from symmetry EHS which corresponds to the element of the Frattini subgroup $\mathcal{EH}$. In fact, if the BCM does not satisfy EHS, then the ensemble of properties it satisfies cannot correspond to any subgroup of order higher than 2 (i.e., the BCM cannot possess more than one symmetry property) and we are allowed to check only symmetry properties corresponding to subgroups of dimension 2, i.e., we can check only symmetries IS, EHIS, ES and HS to eventually conclude that the BCM does not possess any symmetry property.

5 Examples of BCMs that concurrently satisfy feasible combinations of symmetries

Let us now focus our attention to the existence of BCMs satisfying the possible combinations of symmetries specified by Proposition 2: we are wondering if, for each possible subgroup of $(\Sigma, \otimes)$, there exists at least one confirmation measure c that satisfies all the corresponding symmetry properties (and only them) specified by Proposition 2. In other terms, given the 10 equivalence classes individuated by Proposition 2 in the set of BCMs, we want to understand if some of them are perchance empty. In order to do that, we start defining two BCMs that, to the best of our knowledge, were never defined before.
The first BCM we define is

$$EH_2(H, E) = \frac{P(H|E) + P(\neg H|\neg E)}{P(\neg E|H) + P(E|\neg H)} - 1 \; . \tag{1}$$

Interestingly, the measure EH_2 can be recasted in a meaningful way using the language of diagnostic tests. In fact, we can rewrite the numerator $P(H|E)+P(\neg H|\neg E)$ as $PPV+NPV$ where PPV (Positive Predictive Value) denotes the proportion of positive predictions of a statistic test, while NPV (Negative Predictive Value) indicates the proportion of negative predictions of the test. In a similar way, the denominator $P(\neg E|H) + P(E|\neg H)$ can be written as $FNR + FPR$ where FNR (False Negative Rate) and FPR (False Positive Rate) denote the proportions of positive conditions which yield negative test outcomes and, respectively, of negative conditions which yield positive test outcomes. The measure can therefore be rewritten as

$$EH_2(H, E) = \frac{PPV + NPV}{FNR + FPR} - 1 \quad .$$

The second BCM we propose is defined as

$$LEH_2(H,E) = \log \frac{P(H|E) + P(\neg H|\neg E)}{P(\neg E|H) + P(E|\neg H)} \ . \tag{2}$$

The two BCMs are ordinally equivalent, in fact $LEH_2(H,E) = \log(EH_2(H,E)+1)$ but, as we will see, LEH_2 satisfies more symmetry properties than EH_2.
We can now state that for any feasible combination of symmetries there exist a BCM satisfying exactly those properties. More precisely:

Proposition 3 *There exist Bayesian Confirmation Measures that satisfy exactly one of the following sets of concurrent symmetries:*

(i) no symmetry property;

(ii) exactly one symmetry among IS, $EHIS$, EHS, ES and HS;

(iii) exactly $IS \wedge EHS \wedge EHIS$ or $EHS \wedge EIS \wedge HIS$ or $ES \wedge HS \wedge EHS$;

(iv) all the symmetry properties ES, HS, IS, EHS, EIS, HIS, $EHIS$.

Proof
For each possible set of symmetries we provide an example of a BCM that satisfies exactly those symmetries, we avoid here the tedious proof of all the symmetry properties of the BCMs we recall.
i) The confirmation measure $c_1(H,E) = \sqrt{P(E|H)} - \sqrt{P(E)}$ proposed by Greco et al. [17] doesn't satisfy any symmetry property.
ii) Finch's $R(H,E) = P(H|E)/P(H) - 1$, Rips's $G(H,E) = 1 - P(\neg H|E)/P(\neg H)$, Mortimer's $M(H,E) = P(E|H) - P(E)$ and Carnap's $d(H,E) = P(H|E) - P(H)$ (see [3] for more details) satisfy only IS, $EHIS$, ES and HS, respectively (for d and M, symmetries ES, HS and EHS are proved in [17]). The BCM EH_2 defined by (1) satisfies (only) EHS symmetry.
iii) Considering the subsets of 3 symmetries, Cohen's K [6]

$$K(H,E) = \frac{[P(H|E)P(E) + P(\neg H|\neg E)P(\neg E) - P(E)P(H) - P(\neg E)P(\neg H)]}{[1 - P(E)P(H) - P(\neg E)P(\neg H)]}$$

and Nozick's N, $N(H,E) = P(E|H) - P(E|\neg H)$ (see, e.g., [3]) are examples of confirmation measures for which only $\{IS, EHS, EHIS\}$, and $\{ES, HS, EHS\}$, respectively, are met. LEH_2 defined by (2) satisfies (only) $\{EHS, EIS, HIS\}$.
iv) Finally, Carnap's $b(H,E) = P(E \cap H) - P(E)P(H)$ (see [3]) possesses all symmetry properties.

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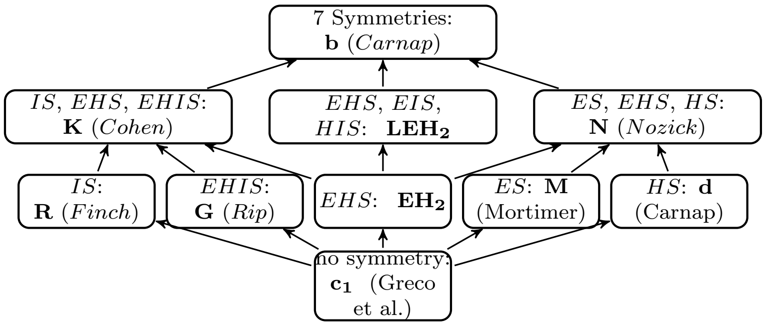

Figure 2: Examples of BCMs that satisfy the feasible combinations of symmetries, embedded in the subgroup lattice structure of $(\Sigma, \otimes)$.

6 Desirability and concurrent symmetry properties of a BCM

Several authors have analyzed the desirability and, conversely, the undesirability of particular symmetry properties of confirmation measures.

Among them, we first recall Eells and Fitelson [9] and Glass [14]; they argued that an *acceptable* BCM should not exhibit symmetries ES, EHS and IS, they proposed HS as the only compelling desideratum while they did not consider $EHIS$, EIS and HIS symmetries at all. Along with Proposition 2, undesirability of EHS necessarily implies that BCMs satisfying the concurrent symmetries in the sets $\{IS, EHS, EHIS\}$, $\{EHS, EIS, HIS\}$, and $\{ES, EHS, HS\}$ should not be considered. At the same time, any BCM for which either just one symmetry among IS, $EHIS$, EHS and ES is met, or, even, if it satisfies all the seven symmetries, should be considered as not adequate. Remark that those authors, implicitly assert that only BCMs that belong to the equivalence class satisfying solely HS hold desirable symmetry properties. Carnap's d is an example of *acceptable* confirmation measure in Eells and Fitelson's framework. Crupi et al. [7] widened the set of the symmetries that an adequate BCM should (or should not) satisfy; within their discussion, they distinguished the cases of confirmation and disconfirmation. In case of confirmation, they argued that only HS, HIS and $EHIS$ can be considered as desirable properties, while all other symmetries should be considered undesirable. In case of disconfirmation, they have proposed HS, EIS and IS as properties that an adequate measure of confirmation should possess, while the other symmetry properties were considered as not desirable. Rescher's Z, defined in [19], is an example of BCM that meets all the symmetry requirements considered desirable in [7], both in case of confirmation and in case of disconfirmation. However, we can observe that, again, only HS was considered a desirable symmetry in both situations in [7].

In the field of rule interestingness, Greco et al. [16] considered ES, EHS and HS

as desirable properties, all the other symmetry properties were classified as undesirable: it is interesting to recognise in their suggestion one of the possible coexistent combinations of symmetries in the subgroup lattice structure of $(\Sigma, \otimes)$, i.e. the Klein subgroup $\{\sigma_0, \mathcal{E}, \mathcal{H}, \mathcal{EH}\}$. Considering the whole list of desirable/undesirable symmetry properties, a good measure appears to be, for example, Nozick's N (see Figure 2), as Greco et al. [16] suggest; but, taking their suggestion in a less restrictive way, one could consider *acceptable* also a BCM that satisfies just one of the symmetries $\{ES\}$ (like Mortimer's M), $\{EHS\}$ (as the above defined EH_2) or $\{HS\}$ (e.g., Carnap's d), hence assuming implicitly a quite larger group of *acceptable* BCMs.

More in general, given different contexts and opinions on which symmetry properties should be satisfied by a confirmation measure, Proposition 2, i.e., the lattice of subgroups of $(\Sigma, \otimes)$, allows to determine quite easily whether (and which) symmetry requirements may coexist or not.

7 Conclusions

Geometric, visual and analytical ways to compare the Bayesian Confirmation Measures have already been suggested (see [3], [20], [21]). In particular, symmetry properties were largely discussed by the literature (see, e.g., [4], [7], [9], [17], [21]).

A group theoretical approach allows a clear insight on symmetry properties, by considering the isomorphism of the group of symmetry functions on sentences pairs (see [7]) and the dihedral group D_8. We observe that another, similar, group theoretic approach was already proposed by Susmaga and Szczęch [21] but referring to contingency tables and making use of the relation of symmetries to the permutation group S_4 of order 24.

In our case, the use of symmetry functions and the isomorphism with the dihedral group of order 8 allows indeed a simpler statement of the concurrent properties of the set of BCM symmetries. In fact, one of the results of the present paper concerns the exploitation of the group structure to detect a representative BCM for each of the 10 equivalence classes determined by the possible sets of symmetries, as determined by the subgroup lattice of the group of symmetry functions. The structure of the subgroup lattice allows also to suggest ways to a faster proof of symmetry properties of a BCM and a clear way to determine whether a BCM possesses desirable properties, where desirability is a concept which is defined by the literature, even if with somewhat conflicting opinions.

In [5] we generalized our proposal to the case of fuzzy confirmation measures (see [13]) and we are currently working on the possible extensions of our approach to the case of BCMs which are defined in different ways in case of confirmation and disconfirmation (see [7]), to the case of weak symmetries (see [18]).

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