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Dmitry Levando and Maxim Sakharov

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Dmitry Levando
National Research University Higher School of Economics, Moscow, Russia

and

Maxim Sakharov
Bauman Mstu, Moscow, Russia

Abstract
We develop a theory of market fluctuations caused by strategic trade with complete information and without outside shocks. The constructed general equilibrium duopoly is a strategic market game with infinite strategies and multiple mixed strategies equilibria. First order conditions (FOC) of the game are the ill-posed problems (Hadamard, 1909), but every equilibrium mixed strategy can be only approximated. This imposes restrictions on convergence of common beliefs of players about actions of each other, existence of rational expectations and a price discovery property of the market, although the market is informationally efficient (Fama, 1970). We suggest a modification of Tikhonov regularization to construct pseudo-solutions. All endogenous variables of the model are exposed to unremovable instabilities, 'natural instabilities', specific to parameters of a chosen approximation. Our result is also related to existence of common knowledge, sun-spot equilibrium, and noise trade.

Keywords
Strategic market games, ill-posed problems, common knowledge, rational expectations, efficient market, price fluctuations

JEL Codes
C68, C61, C72, D59, E31, E32, G14, G7

Address for correspondence:
Dmitry Levando
National Research University Higher School of Economics,
Moscow, Russia
e-mail: dlevando@hse.ru

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Department of Economics
Ca' Foscari University of Venice
Cannaregio 873, Fondamenta San Giobbe
30121 Venice Italy
Fax: ++39 041 2349210

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DMITRY LEVANDO,

NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS,

MAXIM SAKHAROV,

BAUMAN MSTU, MOSCOW, RUSSIA

Abstract. We develop a theory of market fluctuations caused by strategic trade with complete information and without outside shocks. The constructed general equilibrium duopoly is a strategic market game with infinite strategies and multiple mixed strategies equilibria. First order conditions (FOC) of the game are the ill-posed problems (Hadamard, 1909), but every equilibrium mixed strategy can be only approximated. This imposes restrictions on convergence of common beliefs of players about actions of each other, existence of rational expectations and a price discovery property of the market, although the market is informationally efficient (Fama, 1970). We suggest a modification of Tikhonov regularization to construct pseudo-solutions. All endogenous variables of the model are exposed to unremovable instabilities, 'natural instabilities', specific to parameters of a chosen approximation. Our result is also related to existence of common knowledge, sun-spot equilibrium, and noise trade.

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JEL : C 68,61,72; D 59; E31,32; G1 4,17

1. Introduction

Short-run price instability research has a long history: Kendal (1952),[1] Cootner (1964), and later Fama (1970) explained price fluctuations as a reaction to information inflows.[2] Next generation of literature concentrated on microstructure properties of trade with an exogenous stochastic order flows, OHara (1995), Madhavan (2000), Stoll (2003), and Hasbrouck (2007) among others. Empirical instability of prices is registered also for high-frequency trade (HFT) with tick-by-tick transactions, for example, Jarrow and Protters (2012). However, for tiny time intervals, less than $10^{-5}s$ with a tick-by-tick transaction data, information inflow or order flow analysis do not seem to be appropriate tools to explain fluctuations.

Motivated by these observation we suggest a theory of price instability in a general duopolistic equilibrium with complete information and without outside shocks; when the price instability appears from

[1] "there is no hope to predict movements of the exchange for a week ahead without extraneous information", and further conclusion about weakly indices of industrial shares was: " The data behave almost like wandering series...", " it is therefore difficult to distinguish by statistical methods between a genuine wandering series and one wherein the systematic element is weak ... ", " an analysis of stock-exchange movements revealed little serial correlation within series... ", " there is no hope to predict movements of the exchange for a week ahead without extraneous information "

[2] "the only price change would occur . . . from new information, (Cootner, 1964, cited at Fama, 1970)

individually motivated strategic trade.[3] An explicit pricing mechanism allows traders do their best to revert terms of trade in own favor, but a trader has an indeterminacy in beliefs about strategies of another. Multiplicity of mixed strategies equilibria[4] does not let players develop converging equilibrium beliefs, and the price discovery property of the market evaporates.

We suggest a method to approximate equilibrium mixed strategies and an equilibrium price, the method is based on Tikhonov regularization. The resulting price instability can be only approximated, what is suggested to title a 'natural equilibrium instability'. Finally, we demonstrate that information theory approach may not resolve a coordination problem between traders about parameters of regularization.

Imperfect competition as the important research direction for studying financial markets was argued by Stein (2009),[5] "we are converging to a world in which the smart-money players trade intensively with one another". The model used in our paper is a simple economy of two individuals, A and B, with an explicit pricing mechanism. Each individual has some quantity of a consumable good, both supply the total quantity of this good to a market. Both A and B have also a quantity of another consumable good, which is a means of trade; and

[3]Individuals are not 'shock-takers', or 'shock-absorbers', but are direct 'shock-makers' or 'shock-producers'

[4]as an ill-posed problem of mathematics

[5]"Imagine a market in which there are both naive investors with biased expectations and fully rational arbitrageurs. Now let the capital controlled by the latter group grow increasingly large relative to that of the former", and then he asks, "Is it the case that the market is necessarily made more efficient, in the sense that prices on average wind up closer to fundamental values, and non-fundamental sources of volatility become less important?"

an individual has a trade-off: the more one pays, the more one may get from the market, but the more one pays the less consumable money is left. At the same time, a final allocation depends also on a strategy of another: the individuals compete for better terms of trade. How much each shall pay, how shall each play?

Shapley and Shubik (1977) were the first to formalize the described situation as a strategic market game (SMG) with pure strategies. We expand their result for a mixed strategies case, when each player pursues expected utility maximization; a trade is a sell-all version of SMG with a fixed supply and strategic demands. The first order conditions of the problem are the 1-st kind Fredholm integral equations, which can be solved exactly only in rare cases, see Kabanikhin (2011).

Permanent existence of a trivial equilibrium (or no-trade equilibrium) is the standard property of SMG, noted by every author, who wrote about SMG, for example, Shapley and Shubik (1977), Dubey and Shubik (1978). SMG may have multiplicity of pure strategies equilibria, what was demonstrated by Peck and Shell (1985). A finite players SMG *does* differ from a Walrasian competitive outcome, but if a number of participants increases infinitely, then the equilibrium converges to a competitive Walrasian one. Other properties of SMG can be found in surveys of Giraud (2003), and Levando (2012). Our contribution to SMG literature is the method to approximate mixed strategies equilibria and important endogenous economic indicators of the model.

The paper has the following layout. After a relation of our results to existing literature we describe a model, then a numerical method to solve it, finally present a numerical example. Discussion and Conclusion demonstrate additional connections of implications of the model. The Appendix supplies a formal proof for ill-posed property of the game.

2. Related literature

Our paper relates to prior papers in few areas suggesting new insights for each: construction of common knowledge, rational expectations approach, and information efficiency of markets.

Competitive economy cases with outside shocks were introduced by Debreu (1959), and Arrow (1964). Existence of an equilibrium requires players to have converging interactive beliefs, (Aumann, 1999a,b), which are the subject of epistemic game theory, for example, Brandenburger and Dekel (1993), Aumann and Brandenburer (2014), Brandenburger, Keisler, and Jerome (2006), Brandenburger, Friedenberg, and Keisler (2012), Siniscalchi (2016), Battigalli and Siniscalchi, (2002) among others. The converging beliefs operate as an implicit coordination device for players to reach an equilibrium. In an equilibrium of our game converging beliefs can not be constructed explicitly, but only approximated, in many different ways.

Our model is not equivalent to a setting of rational equilibrium approach (for example, Shiller (1978), Barro (1981), Radner (1980), Grossman (1981) among others): players do have complete information about the world and about endowments/payoffs of each other. Different from Muth (1960) (uncertainty about outside shocks), Grossman (1981) and Shell (2008) (rational expectation as an equilibrium device with coordinated beliefs) the trade results in negative externalities and price instabilities, caused purely by strategic actions. Our approach adds to rational expectation literature by imposing a bound of imperfect competition to existence of rational expectations equilibrium. In this way we suggest another reason for "equilibrium degree off disequilibrium", the term of Grossman and Stiglitz, (1980). The same differently, instabilities and fluctuations may not be signals or indicators of a disequilibrium behavior (Samuelson), but are a natural strategic phenomenon.

Recently Shubik and Quindt (2014) wrote about limitation for rational expectations due to sources of consistency between vision of traders on a market, [6] noting at another place : "From the viewpoint of game theory, it[7] appears to be nothing more than a different description of the consistency conditions for the existence of a perfect noncooperative equilibrium." Consistency requires a unique converging belief system; if an individual belief system can not be constructed than individual

[6]" Since the 1970s, economists have increasingly used the concept of rational expectations. Rational expectations is the modeling assumption that agents use forecasting mechanisms that are mutually consistent, given the information available to them. From the viewpoint of game theory, it appears to be nothing more than a different description of the consistency conditions for the existence of a perfect noncooperative equilibrium. From the viewpoint of parallel dynamic programming models of the economy, it provides a mathematical device to patch up needed terminal conditions in such a way as to make learning and the formation of expectations irrelevant to the equilibria being studied. A great and desirable simplification is provided at the cost of accompanying the mathematics with verbal implications that a system not in equilibrium will learn how to achieve the coordination called for by an equilibrium. We have considerable doubts that this finesse in the formulation of expectations provides an adequate description of macroeconomic reality. "

[7]i.e. rational expectations

"forecasting mechanisms" fail to exist, or being constructed becomes mutually inconsistent between traders.

The result of the model predicts that a market can be not a coordination mechanism, as rational expectations theory forecasts; imperfect competition imposes a bound "fully revealing" property of rational expectations. Limits of a market as a coordination device were investigated earlier. For example, Abreu and Brunnermier (2002, 2003) studied mis-coordination and concluded that it can come from individual uncertainty about *when the others* will trade. Stein (2009) himself assigned coordination role to uncertainty about *how many others* will act. But our approach differs from the existing literature in terms of an origin of and structure of uncertainty.

The difference with global games of Shin, Morris and Yilditz (2016) is that they use the very special type of utility function, being substituted into expected utility it generates so called separating kernels. This type of equations has a unique equilibrium, possibly in mixed strategies. Properties of the mixed strategies are determined by a structure of external disturbance along with uniqueness of solutions lead to coordination. In our game players can not construct common beliefs and there are no outside signals, which can be used for coordination.

Our result is also related to recent literature on uncertainty of traders about strategies of others or about information rent of others; this literature is very limited at the moment, Easley, O'Hara and Young (2013), Gao, Song and Wang (2012), a survey is in Banerjee and Green

(2014). Our forecast is in a general case this information can be only imprecise, and resulting price instabilities are unavoidable.

The result has the relation to a voluminous literature on information efficiency of markets and price discovery through trade. Our market satisfies information efficiency property of a market,[8] but prices reflect not a market value, but players' uncertainty about beliefs of each other; prices are unstable not due to properties of an environment of *ex ante* features of traders themselves.

Finally, price discovery, as an evaluation of a good, looses precision. A "normal price" of Marshall acquires a degree of indeterminacy, a perfect foresight of Hicks becomes uncertain. This means that in our model one can not predict prices only from market considerations only.

There is a strand of literature, which focuses on exogenous fluctuations in allocations, calling the results "sunspot equilibrium", (Shell, (1977), Peck and Shell (1985)). Cass and Shell (1983) used the terms "extrinsic uncertainty", "animal spirits ", "market psychology", "sunspots" as synonyms. Prescott and Shell (2002) noted that "the sole purpose of extrinsic uncertainty is to introduce randomization into allocations". Existing literature on sunspot equilibrium is extensive and based essentially on perfect multi-period trade competition, for example, Peck (1975), Pearlman and Sargent (2005), Balasko, Lagos and Wright, (2003), Kajii (1997), Gottardi and Kajii, (1999), Davila, Gottardi and Kajii, (2007), Venditti, Kazuo Nishimura Yannelis (2016),

[8]Fama, (1970): "(i) there are no transaction costs, (ii) all available information is costlessly available to all market participants, (iii) all agree on the implication of current information for current prices ".

Kang (2016). Two periods in the models are required to absorb the first period shocks to obtain a fluctuation in the second period. Another property of these literature is that perfect competition, as a pricing approach, ignores strategic individual decision making.

Goenke and Shell (1997) describe extrinsic uncertainty with an exogenous coordination device in the sense of a correlated equilibrium (also in Aumann, Peck and Shell (1985). All these authors suggest that a coordinating mechanism can be outside or inside the game. Existence of a coordination requires coordination of beliefs, what does not appear in our model.

In the seminal paper on 'noise trade' Kyle (1985) assumed a presence of an informed trader and many uninformed traders, which generate an order flow. In the short run a market maker can not distinguish between informed and uninformed trade. Our approach suggests that to produce market fluctuations one need only a finite number of strategic traders, but not a stochastic order flow.

Our approach is to study strategic market activity without informational advantage about the economy. The only asymmetry is the asymmetry in beliefs about strategies of each other: traders can not overcome indeterminacy over beliefs of each other, what is finally translated into price instability.

3. Sell-all strategic market game

There are two players in a game, $i = 1, 2$. Every player i has positive and finite quantities, Q_i and B_i, of two infinitely divisible endowments; every endowment type is consumable. There is a trading post, which collects offers and bids. Every player forwards to the post the available quantity Q_i, and the market aggregate supply, Q, is $Q = Q_1 + Q_2$; players compete for a share from it. In this way our model is similar to classical fixed supply trade models, but with strategic demands.

The trading post collects bids or payment decisions from traders. Let $E_i = [0; B_i] \subset \mathbb{R}_+$ be a range of payments player i can perform. Let $S_i = \{s_i \equiv b_i \colon s_i \in E_i\}$, be i's set of payments, where b_i is an i's payment decision.

We can use notation E_i for a set of payments (a resource constraint), and S_i for the set of pure strategies, as E_i and S_i coincide numerically.[9] Similarly, b_i is a payment, $b_i \in E_i$, s_i is a strategy, $s_i \in S_i$. Economic and game theory approaches have different interpretations[10] for numerically equivalent variables b_i and s_i. Emphasizing that a payment is a strategic decision we can use s_i instead of b_i.

The trading post fixes explicitly a price[11], as in Shapley and Shubik (1977): a ratio of a total demand $b_1 + b_2$ to the total supply Q:

$$p = \begin{cases} \dfrac{b_1 + b_2}{Q}, & b_1 + b_2 \neq 0 \\ 0, & \text{else} \end{cases}.$$

A player can always submit $s_i = b_i = 0$, and a best response of another player will be $s_{-i} = b_{-i} = 0$, where as usual $-i = \begin{cases} 1, & i = 2 \\ 2, & i = 1 \end{cases}$.

[9] they are isomorphic in mathematical sense, but not in semantical .
[10] semantical and pragmatical
[11] a terms of trade

This is the trivial Nash equilibrium in the game, what disturbs continuity of the best response strategy, and makes the SMG approach be different from the Walrasian approach.

Final partition of Q between players takes place after the market price p is fixed; player i obtains quantity $Q\frac{b_i}{b_1+b_2}$, a share of i's demand in the total demand. The payment b_i is confiscated by the trading post[12], so i has quantity $B_i - b_i$ of the second good left for consumption. Payoff for each player is defined over final allocations of two goods:

$$U_i(b_i, b_{-i}) = \sqrt{\left(Q\frac{b_i}{b_1 + b_2}\right)} + \sqrt{(B_i - b_i)}.$$

The first term is utility from consumption of good 1, the second term is from good 2.

Every player has some impact over the price and exploits own market power strategically. Different from Bertrand analysis is that this is the general equilibrium, not a partial equilibrium analysis. Multiplicity of pure strategies equilibrium (Peck and Shell, 1978) requires switching to mixed strategies equilibrium analysis.

Let $\Delta_i = \left\{\mu_i(b_i) \colon \int_{E_i} \mu_i(b_i) db_i = 1\right\}$ be a set of all mixed strategies of i over the set of pure strategies $S_i \equiv E_i$, the integral is a Lebesgue integral. Mixed strategy $\mu_i(b_i)$ is a probability that i chooses payment b_i. A mixed strategy $\mu_i(b_i)$ can be also addressed as a probability measure over E_i. The set Δ_i includes different types of probability measures: concentrated at points, continuous and their mixtures (Simon

[12]An option is that b_i is reallocated to player $-i$ to make a final allocation $B_{-i} - b_{-i} + b_i$, but this will change nothing in the mathematical structure of the model and conclusions.

and Reed, 1980); it has weak convergence. Δ_i is a topological space of probability measures without a countable base. Expected utility maximization problem of i is standard:

(1)
$$EU_i\left(\mu_i,\mu_{-i}\right) = \max_{\mu_i(b_i)\in\Delta_i(E_i)} \int_{E_i\times E_{-i}} U_i\left(b_i,b_{-i}\right)\mu_i(b_i)\mu_{-i}(b_{-i})db_i db_{-i},$$
$$\text{subject to } \mu_i(b_i)\geq 0, \quad \text{and} \int_{E_i}\mu_i(b_i)db_i = 1,$$

where $\mu_{-i}(b_{-i})$ is a mixed strategy of $-i$, $\int_{S_i}\mu_i(b_i)db_i = 1$ is a normalization condition. As usual, player i takes a mixed strategy of $-i$ as given, $\mu_{-i}(b_{-i})$, and then chooses own best response mixed strategy $\mu_i(b_i)$; player i controls only own mixed strategy $\mu_i(b_i)$, but not other's $\mu_{-i}(b_{-i})$.

Nash equilibrium is a pair of mixed strategies $(\mu_1^\star,\mu_2^\star)$ such that for every $\mu_i\neq\mu_i^\star$ there is

$$EU_i(\mu_i^\star,\mu_{-i}^\star)\geq EU_i(\mu_i,\mu_{-i}^\star), \quad i=1,2.$$

Nash equilibrium exists as a mapping of a bounded, closed continuous set into another one. However, existences does not guarantee that it can be constructed as an exact number. The first order condition of individual expected utility maximization is:

$$\int_{E_{-i}} U_i(b_i, b_{-i})\mu_{-i}(b_{-i})db_{-i} = \lambda_i, \forall b_i \in S_i, \tag{2}$$

$$\text{subject to } \mu_{-i}(b_{-i}) \geq 0, \quad \text{and } \int_{E_{-i}} \mu_{-i}(b_{-i})db_{-i} = 1,$$

where $\lambda_i \neq 0$ is a non-zero Lagrangian multiplier, $\mu_{-i}(b_{-i})$ is unknown probability distribution, or unknown mixed strategy of $-i$. The equation (2) is constructed using of calculus variation. Informally, (2) means that for every pure strategy payment b_i of i, $b_i \in E_i$, the first order condition smoothes individual payoff $U_i(b_i, b_{-i})$ to some non-zero constant; the smoothing happens with mixed strategies of another player. There is only one restriction on λ_i, $\lambda_i \neq 0$.

Mathematical property of our problem is similar to those in geophysics, laser beam scattering (Wang, 2013), in atmospheric optics, etc. The equation (2) has interpretation, met in varied physical and applied measurements (Kozlov, Turchin and Malkovsky, 1971): a device $\int_{E_{-i}} U_i(b_i, b_{-i})db_{-i}$, measures unknown variable $\mu_{-i}(b_{-i})$ for every $\forall b_i \in E_i$ and registers λ_i, which contains an unknown constant and an observation mistake. This takes place for the whole range of observations E_i, $\forall b_i \in E_i$. For different b_i measurement mistakes are samples from some random distribution and are translated into a set of observations Λ_i.

The inverse physical problem is to reconstruct unknown $\mu_{-i}(b_{-i})$ from observations in Λ_i with known properties of the measurement device $\int_{E_{-i}} U_i(b_i, b_{-i})db_{-i}$. Still there is no mathematical theory for approximation of such problems, that guarantees existence and uniqueness of solution, especially with the constraints $\mu_{-i}(b_{-i}) \geq 0$ and $\int_{E_{-i}} \mu_{-i}(b_{-i})db_{-i} = 1$.

The set of payoffs in SMG has a discontinuity at $(b_i, b_{-i}) = (0, 0)$, but for both $i = 1, 2$ the integral operators

$$\int_{E_{-i}} U_i(b_i, b_{-i})db_{-i} < \infty, \quad \forall b_i \in E_i,$$

are bounded and continuous, in the sense of improper Cauchy integrals. Appendix 1 contains a proof that the integral equation (2) does not have an exact solution.

Equation (2) is an ill-posed or improper problem, informally this means that the more precise is observation λ_i, the less precise is a reconstruction for unknown $\mu_i(b_i)$.[13] Kabanichin (2005) is the excellent guide for this kind of problems in different fields of mathematics, Petrov and Sizikov (2011) is the very good practical guide. An instability of solution is the innate property of this kind of problems, that limits our ability to work with exact values of variables.

Condition (2) has the very standard game theory interpretation: in an equilibrium a player is indifferent between equilibrium mixed strategies of another.[14] But from another side, these equilibrium strategies can be known only approximately; an equilibrium can be only approximated, equilibrium strategies are only pseudo-equilibrium strategies,[15]

[13]The situation is similar to indeterminacy in physics.

[14]This means that after a long repetition of the game a player can not identify which pure strategy is was played actually and when. This has the implication for statistical physics concepts for analysis of strategic trade.

[15]or pseudo-solutions of equation (2)

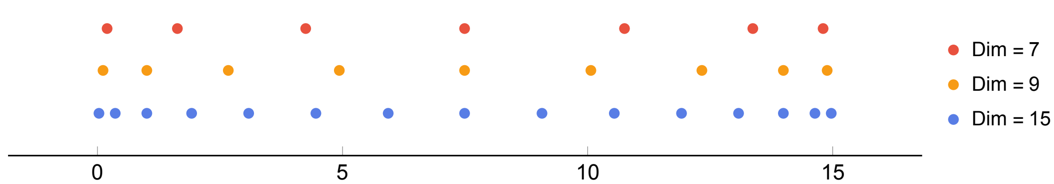
and (2) can be satisfied only approximately, what reproduces properties of physical problems in Kozlov, Turchin, Malkovsky, 1971. In terms of the Nash equilibrium we study an approximate ϵ-Nash equilibrium.

4. Transformation of integral equation into matrix equation

Before we turn to numerical example, we explain an approximation method to construct a numerical pseudo-solution of (2). Let for i there are dim different payments b_{i,r_i}, indexed by $r_i = 1, \ldots, dim$, all belonging to the endowment $E_i = [0, B_i]$.

Approximating points, or collocation points, are nodes of the 1-st kind Tchebychev polynomials. These polynomials provide the best approximation in a class of polynomials with a maximum power dim, their nodes, or collocation points, are more sparse in the middle and more concentrated closer to bounds, to reduce bound effects, for example, see Figure 1.

Figure 1. **Collocation points**



Player i has dim collocation points b_{i,r_i} in E_i, $r_i = 1, \ldots, dim$. Then for every b_{i,r_i} the integral $\int_{S_{-i}} U_i(b_i, b_{-i}) db_{-i}$ can be approximated by a trapezoid formula

$$(3) \quad \int_{E_{-i}} U_i(b_i, b_{-i}) db_{-i} \approx U_i(dim) =$$

$$= \alpha_{r_i,1} U_i(b_{i,r_i}, b_{-i,1}) + \alpha_{r_i,2} U_i(b_{i,r_i}, b_{-i,2}) + \cdots + \alpha_{r_i,dim} U_i(b_{i,r_i}, b_{-i,dim}) = \sum_{\substack{r_{-i}=1 \\ r_i=const}}^{dim} \alpha_{r_i,r_{-i}} U_i(b_{i,r_i}, b_{-i,r_{-i}})$$

$$\text{where } \alpha_{r_1,r_2} = \begin{cases} 1/4, & \text{for } \begin{cases} r_1 = r_2 = 1; r_1 = r_2 = dim \\ r_1 = 1, r_2 = dim; r_1 = dim, r_2 = 1 \end{cases} \\ 1/2, & (r_1 = 1 \text{ or } r_1 = dim) \text{ and } r_2 \neq r_1 \\ 1/2, & (r_2 = 1 \text{ or } r_2 = dim) \text{ and } r_1 \neq r_2 \\ 1, \text{ else} \end{cases}.$$

For all $(b_{i,1}, \ldots, b_{i,dim})$ the integral in the left-hand side of (2) can be approximated by the square matrix:

$$U_i(dim) = \begin{pmatrix} \alpha_{1,1} U_i(b_{i,1}, b_{-i,1}), \ldots \alpha_{1,dim} U_i(b_{i,1}, b_{-i,dim}) \\ \ldots \\ \alpha_{dim,1} U_i(b_{i,dim}, b_{-i,1}), \ldots \alpha_{dim,dim} U_i(b_{i,dim}, b_{-i,dim}) \end{pmatrix}.$$

Every line in matrix $U_i(dim)$ holds b_{i,r_i} constant, every column holds $b_{-i,r_{-i}}$ constant. Indices r_i, r_{-i} of approximating points vary in the range $r_i, r_{-i} = 1, \dots, dim$. Then the first order condition (2) is approximated as

$$\begin{pmatrix} \alpha_{1,1} U_i(b_{i,1}, b_{-i,1}) & \dots & \alpha_{1,1} U_i(b_{i,1}, b_{-i,dim}) \\ \vdots & \vdots & \vdots \\ \alpha_{dim,1} U_i(b_{i,dim}, b_{-i,1}) & \dots & \alpha_{dim,dim} U_i(b_{i,dim}, b_{-i,dim}) \end{pmatrix} \begin{pmatrix} \mu_{-i}(b_{-i,1}) \\ \vdots \\ \mu_{-i}(b_{-i,dim}) \end{pmatrix} = \begin{pmatrix} \lambda_{i,1} \\ \vdots \\ \lambda_{i,dim} \end{pmatrix},$$

where $\lambda_{i,r_i}, r_i = 1, \dots, dim$ are Lagrangian multipliers for different points b_{i,r_i}. The equation (2) assumes that all λ_{i,r_i} are equal, $r_i = 1, dim$, or all measurements are exact. The approximating matrix allows variations in the $(\lambda_{-i,1}, \dots, \lambda_{-i,dim})$. The same equation in the matrix notation has the form

$$U_i(dim) M_{-i} = \Lambda_i,$$

$M_{-i} = (\mu_{-i,1}, \dots, \mu_{-i,dim})'$ is a column vector length dim of unknown pseudo-solutions, $\Lambda_i = (\lambda_{-i,1}, \dots, \lambda_{-i,dim})'$ is a column vector of the right-hand side for the approximation of (2), and $'$ is the matrix transposition operation,.

Stability of a linear approximation of solutions of (2) depends on a number of approximating points. Fadeev (1959) demonstrated that the stability of solution for an approximating matrix equation $y = B\beta$ depends on eigenvalues of the matrix $B'B$,[16] where stability is measured by a conditional number constructed as a ratio of a maximum

[16]In the formula for condition number, next page, we use $\lambda(\cdot, \cdot)$ as an operation of finding eigenvalues, but a Lagrangian multiplier always has only a right bottom index, for example λ_{i,r_i}.

to minimum eigenvalues,[17]

$$cond\left(U_i\left(dim\right)\right)=\sqrt{\left|\frac{\lambda\left(U_i'(dim)U_i(dim),\max\right)}{\lambda\left(U_i'(dim)U_i(dim),\min\right)}\right|}.$$

The property of the ill-posed problem is: an unbounded growth in a number of approximating points generates an unbounded growth in the conditional number, $dim \to \infty \Rightarrow cond\left(U_i(dim)\right) \to \infty$.

Neither existing method guarantees that a solution of the equation (2) is either unique, positive or normalized. Applied physical research has suggested many more or less *ad hoc* and not completely mathematically strict procedures, see the recent lectures of Leyffer (2016) or the survey of Amaran, *et all* (2016).

The initial economic problem transformed into the equilibrium condition (2) has a restriction on values of the unknowns: they can not be negative and must be normalized. To satisfies these conditions we develop a modification of Tikhonov regularization (for example, Kabanihin (2005), Petrov and Sizikov (2011)). Let there is a small regularizing parameter ϵ, $\epsilon > 0$. Regularization smooths an unknown solution M_{-i} in the regularizing equation:

$$(4)\qquad \left(U_i'(dim)U_i(dim)+\epsilon I(dim)\right)M_{-i}=\Lambda_i.$$

Smoothing is understood in the sense that for $\epsilon \to 0$ an approximating solution converges to an exact one. An exact solution can be a pointwise distribution with many such points, locations of these points is

[17]We can take $U_i'(dim)U_i(dim)$ as a covariance matrix. The similar effect is responsible for heteroscedasticity in data analysis.

hard to guess; to identify approximate solutions we use smoothing, i.e. regularization.

We follow the method closed to statistical physics simulation for ill-posed problems (Kozlov, Turchin, Malkovsky, 1971), with enforced assigning zero valuers to unfeasible solutions, combined with Monte-Carlo simulation, close examples are in Wang (2013).

Using conditional number we choose parameters ϵ and dim. Then we assign uniform random values from $[0, 1]$ to a column vector Λ_i. These random variables serve the same role as a mistake in physical observations. Uniform distribution is used to avoid discrimination over a range E_i. We do not relate ϵ and the upper bound of the uniform distribution.

An approximating solution (or a pseudo-solution) is constructed by minimizing a discrepency (Petrov and Sizikov, 2005) following Tikhonov regularization and solving a system of equations (4):

$$M_{-i} = \Big(U_i'(dim)U_i(dim) + \epsilon I(dim)\Big)^{-1} U_i'(dim)\Lambda_i.$$

Negative points in obtained vector M_{-i} are assigned zero values,

$$\tilde{\mu}_{i,r_i} = \begin{cases} \mu_{i,r_i}, & \mu_{i,r_i} \geq 0 \\ 0, & \mu_{i,r_i} < 0 \end{cases}, \qquad \hat{\mu}_{i,r_i} = \frac{\tilde{\mu}_{i,r_i}}{\sum_{r_i=1}^{dim} \tilde{\mu}_{i,r_i}}$$

and $\hat{M}_{-i} = (\hat{\mu}_{i,1}, \ldots, \hat{\mu}_{i,dim}) \geq 0$ is a vector of possible normalized solutions, such that $\sum_{r_i=1}^{dim} \hat{\mu}_{i,r_i} = 1$. Enforced assignment can only decrease mistakes in the right hand side of the constructed matrix equation.

Smoothing property of the vector $\hat{M}_{-i}$ is checked by a new approximation

$$U_i(dim)\hat{M}_{-i} = \hat{\Lambda}_i,$$

where $\bar{\Lambda}_i = \left(\hat{\lambda}_{i,1}, \ldots, \hat{\lambda}_{i,dim}\right)$ is an approximation for Lagrangian multiplier λ_i for $\hat{M}_{-i}$.

This procedure is repeated as Monte-Carlo simulation. Results of the simulation are labeled $\bar{M}_{-i}$, an approximated solution, and $\bar{\Lambda}_i$, an approximation for the precision of the solution. Constructed $\hat{M}_i$, $i = 1, 2$ are the pseudo-solutions for mixed strategies of both players. Finally we use these results to construct approximations for prices

5. Numerical characterization of a pseudo-solutions

Now we are ready to present numerical examples with endowment parameters as they are in Shapley and Shubik (1977) $Q_1 = 30$, $B_1 = 15$, $Q_2 = 15$, $B_2 = 30$.

5.1. **Characterization of ill-posed property.** Ill-posed property of a matrix is characterized by the very big conditional numbers in the first row of Table 1: relatively modest numbers of approximating points dim without regularization, $\epsilon = 0$, explode values of conditional numbers. Standard solution methods for systems of linear equations can not be used here. Small positive $\epsilon > 0$ smooth unknown solutions, what is demonstrated by the significant drop in the condition numbers in every columns of Table 1.

	$dim = 7$	$dim = 11$	$dim = 15$	$dim = 25$	$dim = 35$
$\epsilon = 0$	5.481×10^{10} 9.755×10^{10}	2.070×10^{11} 1.690×10^{11}	6.478×10^{11} 1.345×10^{12}	3.250×10^{13} 6.745×10^{11}	1.049×10^{12} 2.039×10^{12}
$\epsilon = \frac{1}{1000}$	1.278×10^{6} 2.373×10^{6}	3.787×10^{6} 6.942×10^{6}	7.620×10^{6} 1.386×10^{7}	2.298×10^{7} 4.145×10^{7}	4.660×10^{7} 8.371×10^{7}
$\epsilon = \frac{1}{500}$	639017. 1.187×10^{6}	1.894×10^{6} 3.471×10^{6}	3.810×10^{6} 6.932×10^{6}	1.149×10^{7} 2.073×10^{7}	2.330×10^{7} 4.185×10^{7}
$\epsilon = \frac{1}{100}$	127804 237354	378754 694199	762010. 1.389×10^{6}	2.298×10^{6} 4.145×10^{6}	4.660×10^{6} 8.371×10^{6}
$\epsilon = \frac{1}{50}$	63902.6 118678.	189377. 347100.	381006. 693247.	1.149×10^{6} 2.073×10^{6}	2.330×10^{6} 4.185×10^{6}

TABLE 1. An example of conditional numbers for two players

	dim=7	dim=9	dim=11
$\frac{1}{4}$	5113.13 9495.14	9469.91 17452.5	15151.1 27768.9
$\frac{5}{8}$	2045.85 3798.65	3788.57 6981.62	6061.04 11108.2
$\frac{3}{4}$	1705.04 3165.71	3157.3 5818.18	5051.04 9256.97

TABLE 2. Additional conditional values. Approximation is done for parameters of the last column, $dim = 11$, $\epsilon = 1/4, 5/8, 3/4$

.

Petrov and Sizikov (2005) recommended to use parameters, where conditional number is no greater than 10^4, for our mixed strategies approximation we take parameters from the last column of Table 2, $dim = 11$ and $\epsilon = 1/4, 5/8, 3/4$.

Table 3 presents our results with Monte-Carlo simulation with 200 repetitions. Every column characterizes a player; the first row are pseudo-solutions or approximations of mixed strategies, the second rows contains the constructed Langrangian multipliers. According to

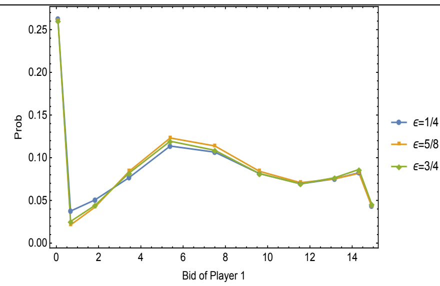
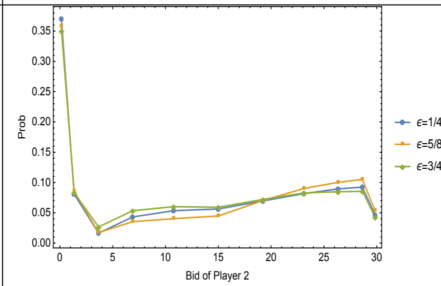
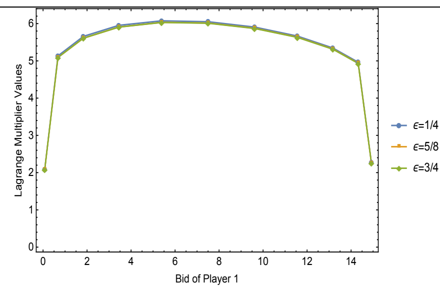
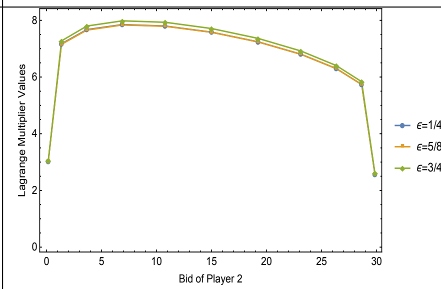
	player 1	player 2
mixed strategy $\mu_i(b_i)$		
Lagrangian multiplier $\lambda_i(b_i)$		

TABLE 3. Simulation of the first order condition $\int_{E_{-i}} U_i\left(b_i, b_{-i}\right)\mu_{-i}(b_{-i})db_{-i} = \lambda_i$

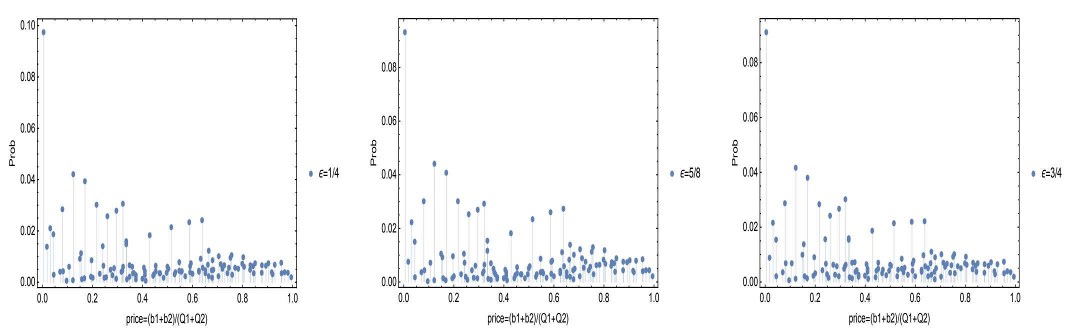
the structure of the first order condition (2) if a solution is in one column, then a corresponding Lagrangian multiplier is in another. Every graph contains jumps (bound effects) in points next to bounds. Formally we can remove the strategies for $b_{i,1}$ and $b_{i,dim}$, but they are the unremovable cost of discreet approximation, in this or that way they will appear for any set of parameters dim and ϵ.

At the same time central parts of all graphs are far from being constant. This is the expected numerical outcome from ill-posed property of expected utility maximization.

The first order condition (2) assumes that all λ_is are all constant or approximately constant. It is impossible to obtain an exact solution for an ill-posed problem, but we can compare smoothing for players. For $i = 1$ the quality of smoothing in a central part of the graph is relatively better than one for $i = 2$. We do not claim that the approximation is the best, we only demonstrate the effect and suggest a tool for numerical analysis. Due to properties of discretization bound effects can not be completely eliminated too.

Usually individual actions are not observable, but prices are public. In Table 4 we present prices constructed from combinations of all mixed strategies of both players. The horizontal axes is a price, the vertical one is probability axes. We can easily see that probabilities of prices are very unstable; they are not concentrated at one point. If a trade is repeated, then different prices will be realized. This very phenomenon we suggest to call 'natural equilibrium instability of prices'. A smoothing effect of regularization is observed in reduction in price volatility with an increase of regularization parameter ϵ from $\epsilon = 1/4$ to $\epsilon = 3/4$, what is better seen in the range of the vertical axes.

Table 4. **Constructed prices**



5.2. **Information content of mixed strategies.** There is a problem of a choice of an equilibrium mixed strategy not yet investigated in existing literature. If every player has many equilibrium mixed strategies, what could be a motivation to choose one. A choice of a strategy is described by two parameters, dim and ϵ. Conditional numbers impose restrictions on these choices.

But there is another constraint, related to individual motivation to hide information about chosen distribution of trading strategies. Similar issues appear in information theory and cryptography: how to hide information in stochastic process. Our prime suggestion is to use entropy as a criterion for information content of mixed strategies. An entropy of a mixed strategy after Shennon[18] can be measured as

$$EN_i = \sum_i prob_i \log(prob_i),$$

.

We justify this choice that i wants to hide a chosen equilibrium mixed strategy by minimizing information content of her trade, where information is measure as $IN_i = \frac{1}{EN_i}$, what is equivalent to entropy maximization. This means that a trading player wants to supply as less information as possible with own mixed strategies.

From a market view, we may think that the maximum entropy approach may induce coordination in parameters, however, this is not true what can be seen from Table 5: players $i = 1, 2$ have different sets of parameters dim and ϵ, which maximize entropy.

[18]there are other definitions of entropy

	ϵ=1/4	ϵ=5/8	ϵ=3/4
Entropy for $i = 1$	2.3	2.27	2.27
Entropy for $i = 2$	2.08	2.06	2.11

Table 5. Comparison of information content of constructed mixed strategies with Shenon entropy

Now we have two inconsistent visions of a market, argued by Shubik and Quindt (2014): the first is caused by multiplicity of parameters for strategic trade, the second is individual choice to hide information content about trade.

6. Discussion

Our goal was to provide rational explanation for market fluctuations happening without visible reasons. Having complete information about environment a market participant may still have uncertainty about trading strategies of others, what is first translated into her trade, and then revealed in observed market prices. The conclusion of our model is that this uncertainty can not be easily removed from the market. 'Natural equilibrium instability of prices' is considered as a non-perfect general equilibrium phenomenon.[19] The constructed instability is the innate property of the mathematical object used for individual expected utility maximization.

The model demonstrates that seemingly disequilibrium behavior (Samuelson, 1965) can appear from strategic behavior. The result is also different from tatônement process: prices do not converge and do not have a nature of adjustment to outside shocks, they are products of non-coordinated strategic actions. Competing traders have some market

[19] "random walk"

power for a better terms of trade, but with uncertainty about trading strategies of others.

'Natural equilibrium instability of prices' is a market phenomenon consistent with a one-shot game. Lucas (1980) argued that market "uncertainty can not be incorporated into static general equilibrium". Our result is that a specially designed simultaneous game can demonstrate this instability for a one period model.

Our analysis offers the interesting link between non-perfect competition, and methods of statistical physics and information theory. From one side, in a mixed strategies equilibrium a player is indifferent between equilibrium (mixed) strategies of another[20]. From another, it is impossible to calculate equilibrium strategies exactly. These properties are close to those observed in applications of statistical physics to ill-posed problems (Kozlov, Turchin, Malkovsky, 1972). From another side, if there is a multiplicity of equilibrium mixed strategies a trader would like to choose one, which supplies the less information the possible; making a mixed strategy a tool to hide private information, what relates our approach to information content of information.

If to reconstruct out game as a repetitive game, with identical endowments in every period, price instability will occur as a stochastic time series: it can be misinterpreted as an inflow of information and make traders change their decisions. For a market with outside shocks this result was described by (Shleifer and Summers, 1990) with a market with not fully rational agents. They may consider price instability

[20]Mixed strategy of one smoothes payoffs profile of another.

as informative signals, what may results in “trend chasing”, although the information is “unwarranted by fundamentals.”

Song, Tan and Wu (2005)[21], Brogaard (2010) argued that, for US market “HFTs tend to follow a price reversal strategy driven by order imbalances”, what means that market price is unstable even if there is no inflow of information. More of that, “HFTs do not seem to increase volatility and may in fact reduce it.”. From another side, (Zhang, 2010) provided evidence that “high-frequency trading is positively correlated with stock price volatility ”. Recent research of high-frequency trade demonstrated that fluctuations have an accumulation effect, impact on volatility, (Andersen, Bollerslev, Diebold, and Labys (2000)) and have an adverse effect for price discovery and for precision of inference about the efficient price. The implication of our model is that these observations correlation can be caused by misperception of price instability by traders.

De-noising data was studied in voluminous literature, which assumes additive errors and some kind of independence in successive observations; the literature includes applications of different statistical methods, for example, Zhang, Mykland, and At-Sahalia (2005), Zhang (2006), Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008), Jacod, Li, Mykland, Podolskij, and Vetter (2009) and Podolskij and Vetter (2009), Xiu (2010). Statistical properties of the high frequency noise are addressed in Jacod, Li and Zheng (2017). Our conclusion that a market

[21]reported for Chinese market that “The number of trades explains the volatility/volume relation better than the size of trades’.’

noise is the innate property induced by strategic trading, and need to deal with ill-posed statistics methods (REFERENCE).

In our next paper we will apply the constructed framework for money market and demonstrate that the same mechanism can explain instability in interactions between monetary and real sectors.

7. Conclusion

In this paper, we have developed a mixed strategy extension of strategic market game of Shapley-Shubik (1977). The numerical approximation of the model is able to demonstrate that at least part of seemingly unreasonable market fluctuations can be caused by strategic trade, which takes place without outside shocks and complete information of players about endowments and payoffs. Uncertainty appears from indeterminacy of beliefs of players about each other.

The model is based on a numerical approximation of an ill-posed problem of the 1-st kind integral Fredholm equation. We suggest an approximation method based on Tikhonov regularization, similar to methods of numerical solutions of inverse problems in physics and engineering.

Our result has implications for few other areas of economic analysis, where the same ill-posed problem appears. For example, epistemic game theory can have cases, where common beliefs/common knowledge can not be constructed exactly. Rational expectations equilibrium for strategic trade may not exist as players do not have converging common beliefs. Finally, all these results induce the evaporation of price discovery property of a market.

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8. Appendix

A proof of ill-posed property of the first order condition 5, repeated below:

$$\int_{E_{-i}} U_i(b_i, b_{-i})\mu_{-i}(b_{-i})db_{-i} = \lambda_i, \quad \forall b_i \in E_i, \tag{5}$$

is rewritten in an operator form

$$K_i \mu{-}i = \lambda_i,$$

where K_i is a linear operator, $\mu{-}i$ is unknown probability distribution, $\lambda_i \neq 0$.

Our proof follows Miller (1974). Let K_i is expanded using singular values functions:

$$K(b_i, b_{-i}) = \sum_j^{\infty} \kappa_j u_j(b_i) v_j(b_{-i}),$$

where $u_j(b_i)$ and $v_j(b_{-i})$ are eigenfunctions and κ_j are eigenvalues. Letter j is an index of the expansion. This requires that

$$\lim_{n\to\infty} \int_{E_i \times E_{-i}} \Big(K_i(b_i, b_{-i}) \kappa_j u_j(b_i) v_j(b_{-i}) \Big)^2 db_i db_{-i} < \infty.$$

The expansions exists due to existence of an upper bound of K_i in the sense of Cauchy and existence of the weak limit in $\Delta_{-i}i$, $\mu_{-i} \in \Delta_{-i}$.

The sequences $\{u_j(b_i)\}$ and $\{v_j(b_{-i})\}$ are orthonormal with the properties: $K_i v_j = \kappa_j u_j$ and $K_i^T u_j = \kappa_j v_j$. Hense $K_i K_i^T u_j = \kappa_j^2 u_j$ and $K_i^T K_i v_j = \kappa_j^2 v_j$, and the the functions $u_j(b_i)$ and $v_j(b_i)$ are eigenfunctions for the symmetric and self-adjoint operators $K_i K_i^T$ and $K_i^T K_i$ respectively for the eigenvalues κ_i^2. Thus the effect of the operator K_i on function μ_{-i} is

$$K_i \mu_{-i} = \sum_j^{\infty} \kappa_j f_j u_j(b_i),$$

where a coefficient f_j is a constructed as

$$f_j = \langle \mu_{-i}, v_j \rangle = \int_{S_{-i}} \mu_{-i}(b_{-i}) v_j(b_{-i}) db_{-i},$$

where notation $\langle \cdot, \cdot \rangle$ is for scalar product.

Lagrangian multiplier is any non-zero number, and let one such multiplier λ_i be expanded with a sequence of orthonormal functions $\{g_j\}$ is

$$\lambda_i = \sum_{j=1}^{N} g_j u_j(b_i) db_i,$$

where g_j is an expansion coefficient defined as

$$g_j = \langle \lambda_i, u_j \rangle = \int_{S_i} u_j(b_i) db_i.$$

This expansion always can be done.

If the operator equation $K\mu_{-i} = \lambda_i$ is to be satisfied then the two expansions $K\mu_{-i} = \sum_{j=1}^{\infty} \kappa_j f_j u_j(b_i)$ and $\lambda_i = \sum_{j=1}^{\infty} g_j u_j(b_i)$, from where follows:

$$\kappa_j f_j = g_j.$$

Let a solution be $\mu_{-i} = \sum_{j=1}^{\infty} \frac{g_j}{\kappa_j} v_j(b_{-i})$, and this is a unique solution.

Is this expansion unique? The answer consists of two pars.

Compatibility.: For any function u_j, such that $K^T u_j = 0$ there is $\langle \lambda_j, u_j \rangle = 0$, so $\kappa_j = 0$ and $g_j = 0$

Convergence.: Existence of a solution in a metric space L_2 requires that $\sum_j^{\infty} (g_j/\kappa_j)^2 < \infty$. This is a "smoothness " condition, the more κ decreases, the more severely is g restricted.

Uniqueness.: Suppose there is a function v_j such that $K_i v_j = 0$. Then if μ_{-i} is a solution, so is $\mu_{-i} + C v_j$, and any solutions differ by a function v_j with this property. Thus uniqueness occurs if and only if the equation $K_i v_j = 0$.

The first two are necessary and sufficient conditions for existence of solution in metric space.

The most important property for the current paper is instability in the solution.

Let a solution μ_{-i} exists and it is unique. Then by Hilbert-Schmidt theory the sequence $\{\kappa_j\}$ is infinite and converges to zero. Let j be big enough and choose κ_j very small. We can choose any $\lambda_i \neq 0$, and make a perturbation for λ_i in a coefficient δg_j by an amount δg_j. This causes a perturbation in the solution $\delta \mu_{-i} = \frac{\delta g_j}{\kappa_j} v_j(b_{-i})$. Then the ratio $\frac{\|\delta \mu_{-i}\|}{\delta \lambda_i}$ can be made arbitrarily large

This means that even we find some approximation for μ_{-i} and normalize it, then we will still have the same problem, normalization procedure will be also unstable, more in Lavrentiev, Turchin, Malkevich (1970).